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Negative Polarity Items

Syntax and Semantics of a Higher Order Logic

Types All expressions of our higher-order language, $\mathcal{L}_{\mathsf{Type}}$, will be typed. To keep the language simple, we will only use two basic types, e (for the basic entities in the domain) and t (for the truth values 0 and 1).

Definition 1 Types
Type *is the smallest set such that*

i.) $e \in \mathsf{Type}$,

ii.) $t \in \mathsf{Type}$,

iii.) *for each* $\tau_1 \in \mathsf{Type}$, *for each* $\tau_2 \in \mathsf{Type}$, $\langle \tau_1, \tau_2 \rangle \in \mathsf{Type}$.

Syntax The basic expressions of $\mathcal{L}_{\mathsf{Type}}$ consist only of variables and constants. In contrast to first order logic there is no distinction between terms and formulae.

Definition 2 Basic Expressions

i.) *For each* $\tau \in \mathsf{Type}$, Var_τ *is the smallest set such that for each* $n \in \mathbb{N}_0$, $v_{n,\tau} \in \mathsf{Var}_\tau$.

ii.) *For each* $\tau \in \mathsf{Type}$, Const_τ *is the smallest set such that for each* $n \in \mathbb{N}_0$, $c_{n,\tau} \in \mathsf{Const}$.

We write Var for the set of all variables, $\bigcup\limits_{\tau \in \mathsf{Type}} \mathsf{Var}_\tau$, and Const for the set of all constants, $\bigcup\limits_{\tau \in \mathsf{Type}} \mathsf{Const}_\tau$.
The set of *basic expressions* of our language is the union of the set of variables and the set of constants.

Definition 3 Meaningful Expressions
The meaningful expressions *of* $\mathcal{L}_{\mathsf{Type}}$ *are the smallest familiy* $(\mathsf{ME}_\tau)_{\tau \in \mathsf{Type}}$ *such that*

i.) *for each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each variable* $v_{n,\tau} \in \mathsf{Var}_\tau$, $v_{n,\tau} \in \mathsf{ME}_\tau$;

ii.) *for each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each constant* $c_{n,\tau} \in \mathsf{Const}_\tau$, $c_{n,\tau} \in \mathsf{ME}_\tau$;

iii.) *for each* $\tau \in \mathsf{Type}$*, for each* $\phi_\tau \in \mathsf{ME}_\tau$*, for each* $\psi_\tau \in \mathsf{ME}_\tau$*,*
$(\phi_\tau \equiv \psi_\tau)_t \in \mathsf{ME}_t$*;*

iv.) *for each* $\phi_{\langle\tau_2,\tau_1\rangle} \in \mathsf{ME}_{\langle\tau_2,\tau_1\rangle}$*, for each* $\psi_{\tau_2} \in \mathsf{ME}_{\tau_2}$*,*
$\left(\phi_{\langle\tau_2,\tau_1\rangle}\,(\psi_{\tau_2})\right)_{\tau_1} \in \mathsf{ME}_{\tau_1}$*;*

v.) *for each* $\phi_t \in \mathsf{ME}_t$*,*
$(\neg\phi_t)_t \in \mathsf{ME}_t$*;*

vi.) *for each* $\phi_t \in \mathsf{ME}_t$*, for each* $\psi_t \in \mathsf{ME}_t$*,*
$(\phi_t \wedge \psi_t)_t \in \mathsf{ME}_t$*;*

vii.) *for each* $\phi_t \in \mathsf{ME}_t$*, for each* $\psi_t \in \mathsf{ME}_t$*,*
$(\phi_t \vee \psi_t)_t \in \mathsf{ME}_t$*;*

viii.) *for each* $\phi_t \in \mathsf{ME}_t$*, for each* $\psi_t \in \mathsf{ME}_t$*,*
$(\phi_t \rightarrow \psi_t)_t \in \mathsf{ME}_t$*;*

ix.) *for each* $\phi_t \in \mathsf{ME}_t$*, for each* $\psi_t \in \mathsf{ME}_t$*,*
$(\phi_t \leftrightarrow \psi_t)_t \in \mathsf{ME}_t$*;*

x.) *for each* $\tau_1 \in \mathsf{Type}$*, for each* $\tau_2 \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each* $v_{n,\tau_2} \in \mathsf{Var}$*, for each* $\phi_{\tau_1} \in \mathsf{ME}_{\tau_1}$*,*
$(\lambda v_{n,\tau_2}.\phi_{\tau_1})_{\langle\tau_2,\tau_1\rangle} \in \mathsf{ME}_{\langle\tau_2,\tau_1\rangle}$*;*

xi.) *for each* $\tau \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each* $v_{n,\tau} \in \mathsf{Var}$*, for each* $\phi_t \in \mathsf{ME}_t$*,*
$(\forall v_{n,\tau}\ \phi_t)_t \in \mathsf{ME}_t$*;*

xii.) *for each* $\tau \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each* $v_{n,\tau} \in \mathsf{Var}$*, for each* $\phi_t \in \mathsf{ME}_t$*,*
$(\exists v_{n,\tau}\ \phi_t)_t \in \mathsf{ME}_t$*.*

Semantics D_e is a set of entities, and $\mathsf{D}_t = \{0,1\}$. For each $\tau_1 \in \mathsf{Type}$, for each $\tau_2 \in \mathsf{Type}$, $\mathsf{D}_{\langle\tau_1,\tau_2\rangle} = {\mathsf{D}_{\tau_2}}^{\mathsf{D}_{\tau_1}}$ (the set of all functions from D_{τ_1} to D_{τ_2}).
Let I be a function assigning a denotation to each non-logical constant, $c_{n,\tau}$, of $\mathcal{L}_{\mathsf{Type}}$ from the set D_τ.
Let $\mathsf{M} = \langle \mathsf{D_e}, \mathsf{I}\rangle$. We will call each M a *model*.
Let g be a function in $\bigcup_{\tau\in\mathsf{Type}} \left({\mathsf{D}_\tau}^{\mathsf{Var}_\tau}\right)$ which assigns an object (of the appropriate type) in the domain $\bigcup_{\tau\in\mathsf{Type}} \mathsf{D}_\tau$ to each variable in Var. We call each g an *assignment function*.
Assume that v is a variable of type τ and d is an element of D_τ. We will use the notation $g\overset{d}{v}$ for the assignment function g' which differs from the assignment function g in the following way:
For each $\tau \in \mathsf{Type}$, for each $v \in \mathsf{Var}_\tau$, for each $x \in \mathsf{Var}_\tau$, for each $d \in \mathsf{D}_\tau$,
$$g\overset{d}{v}(x) = \begin{cases} d & \text{if } x = v \text{, and} \\ g(x) & \text{otherwise.} \end{cases}$$

Definition 4 Denotation
Let M *be a model and* g *an assignment function.*

i.) *For each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each variable* $v_{n,\tau} \in \mathsf{Var}_\tau$,
$[\![v_{n,\tau}]\!]^{\mathsf{M},g} = g\left(v_{n,\tau}\right)$.

ii.) *For each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each constant* $c_{n,\tau} \in \mathsf{Const}_\tau$,
$[\![c_{n,\tau}]\!]^{\mathsf{M},g} = \mathsf{I}\left(c_{n,\tau}\right)$.

iii.) *For each* $\tau \in \mathsf{Type}$, *for each* $\phi_\tau \in \mathsf{ME}_\tau$, *for each* $\psi_\tau \in \mathsf{ME}_\tau$,
$[\![\left(\phi_\tau \equiv \psi_\tau\right)_t]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\phi_\tau]\!]^{\mathsf{M},g} = [\![\psi_\tau]\!]^{\mathsf{M},g}$.

iv.) *For each* $\phi_{\langle\tau_2,\tau_1\rangle} \in \mathsf{ME}_{\langle\tau_2,\tau_1\rangle}$, *for each* $\psi_{\tau_2} \in \mathsf{ME}_{\tau_2}$,
$[\![\left(\phi_{\langle\tau_2,\tau_1\rangle}\left(\psi_{\tau_2}\right)\right)_{\tau_1}]\!]^{\mathsf{M},g} = [\![\phi_{\langle\tau_2,\tau_1\rangle}]\!]^{\mathsf{M},g}\left([\![\psi_{\tau_2}]\!]^{\mathsf{M},g}\right)$.

v.) *For each* $\phi_t \in \mathsf{ME}_t$,
$[\![\left(\neg\phi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\left(\phi_t\right)_t]\!]^{\mathsf{M},g} = 0$

vi.) *For each* $\phi_t \in \mathsf{ME}_t$, *for each* $\psi_t \in \mathsf{ME}_t$,
$[\![\left(\phi_t \wedge \psi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\phi_t]\!]^{\mathsf{M},g} = 1$ *and* $[\![\psi_t]\!]^{\mathsf{M},g} = 1$.

vii.) *For each* $\phi_t \in \mathsf{ME}_t$, *for each* $\psi_t \in \mathsf{ME}_t$,
$[\![\left(\phi_t \vee \psi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\phi_t]\!]^{\mathsf{M},g} = 1$ *or* $[\![\psi_t]\!]^{\mathsf{M},g} = 1$.

viii.) *For each* $\phi_t \in \mathsf{ME}_t$, *for each* $\psi_t \in \mathsf{ME}_t$,
$[\![\left(\phi_t \rightarrow \psi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\phi_t]\!]^{\mathsf{M},g} = 0$ *or* $[\![\psi_t]\!]^{\mathsf{M},g} = 1$.

ix.) *For each* $\phi_t \in \mathsf{ME}_t$, *for each* $\psi_t \in \mathsf{ME}_t$,
$[\![\left(\phi_t \leftrightarrow \psi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\phi_t]\!]^{\mathsf{M},g} = [\![\psi_t]\!]^{\mathsf{M},g}$.

x.) *For each* $\tau_1 \in \mathsf{Type}$, *for each* $\tau_2 \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each* $v_{n,\tau_2} \in \mathsf{Var}$, *for each* $\phi_{\tau_1} \in \mathsf{ME}_{\tau_1}$,
$[\![\left(\lambda v_{n,\tau_2}.\phi_{\tau_1}\right)_{\langle\tau_2,\tau_1\rangle}]\!]^{\mathsf{M},g}$ *is that function* h *from* D_{τ_2} *to* D_{τ_1} *such that for each* $o \in \mathsf{D}_{\tau_2}$, $h(o) = [\![\phi_{\tau_1}]\!]^{\mathsf{M},g\overset{o}{v}_{n,\tau_2}}$.

xi.) *For each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each* $v_{n,\tau} \in \mathsf{Var}$, *for each* $\phi_t \in \mathsf{ME}_t$,
$[\![\left(\forall v_{n,\tau}\ \phi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff for each* $o \in \mathsf{D}_\tau$, $[\![\phi_t]\!]^{\mathsf{M},g\overset{o}{v}_{n,\tau}} = 1$.

xii.) *For each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each* $v_{n,\tau} \in \mathsf{Var}$, *for each* $\phi_t \in \mathsf{ME}_t$,
$[\![\left(\exists v_{n,\tau}\ \phi_t\right)_t]\!]^{\mathsf{M},g} = 1$ *iff for at least one* $o \in \mathsf{D}_\tau$, $[\![\phi_t]\!]^{\mathsf{M},g\overset{o}{v}_{n,\tau}} = 1$.

Standard results tell us that the entire language $\mathcal{L}_{\mathsf{Type}}$ can be given in terms of the clauses *i.)–iv.)* and *x.)* of our syntax and semantics. This means that the logical connectives in *v.)–ix.)* and the quantifiers in *xi.)* and *xii.)* can be defined using these five clauses.