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## Underspecified Semantics: An HPSG Perspective

### Syntax and Semantics of First Order Logic

**Syntax** Let  $\text{Var}$ ,  $\text{Const}$ ,  $\text{Func}$  and  $\text{Rel}$  be at most countably infinite and pairwise disjoint sets of symbols. Each symbol in  $\text{Func}$  and in  $\text{Rel}$  is assigned a natural number called the *arity* of the symbol.

To simplify the formulations in the definitions of the syntax and semantics, we will henceforth assume that the sets  $\text{Var}$ ,  $\text{Const}$ ,  $\text{Func}$  and  $\text{Rel}$  are fixed.

#### Definition 1 Terms

- i.) For every  $v \in \text{Var}$ ,  $v$  is a term.
- ii.) For every  $c \in \text{Const}$ ,  $c$  is a term.
- iii.) For every term  $t_1, \dots$ , for every term  $t_n$ , for every  $n$ -ary function symbol  $F$ ,  $F \in \text{Func}$ ,  $F(t_1, \dots, t_n)$  is a term.
- iv.) Only that which can be generated by the clauses i.)–iii.) in a finite number of steps is a term.

#### Definition 2 Formulae

- i.) For every term  $t_1$ , for every term  $t_2$ ,  $t_1 \equiv t_2$  is a formula.
- ii.) For every term  $t_1, \dots$ , for every term  $t_n$ , for every  $n$ -ary relation symbol  $R$ ,  $R \in \text{Rel}$ ,  $R(t_1, \dots, t_n)$  is a formula.
- iii.) For every formula  $\phi$ ,  $\neg\phi$  is a formula.
- iv.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,  $(\phi_1 \wedge \phi_2)$  is a formula.
- v.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,  $(\phi_1 \vee \phi_2)$  is a formula.
- vi.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,  $(\phi_1 \rightarrow \phi_2)$  is a formula.
- vii.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,  $(\phi_1 \leftrightarrow \phi_2)$  is a formula.
- viii.) For every  $v \in \text{Var}$ , for every formula  $\phi$ ,  $\forall v \phi$  is a formula.
- ix.) For every  $v \in \text{Var}$ , for every formula  $\phi$ ,  $\exists v \phi$  is a formula.
- x.) Only that which can be generated by the clauses i.)–ix.) in a finite number of steps is a formula.

On the basis of the syntactic form of first order formulae we can say what it means for a variable to occur *free* in an expression.

To make this precise, we first define a function,  $\text{var}$ , which assigns to each term the set of variables in it. Then we define a function,  $\text{free}$ , which assigns to each formula  $\phi$  the set of variables which occur free in  $\phi$ .

**Definition 3**  $\text{var}$

$\text{var}$  is the total function from the set of terms to the powerset of  $\text{Var}$  such that:

- i.) For every  $v \in \text{Var}$ ,  $\text{var}(v) = \{v\}$ .
- ii.) For every  $c \in \text{Const}$ ,  $\text{var}(c) = \emptyset$ .
- iii.) For every term  $t_1, \dots$ , for every term  $t_n$ , for every  $n$ -ary function symbol  $F$ ,  $F \in \text{Func}$ ,  

$$\text{var}(F(t_1, \dots, t_n)) = \text{var}(t_1) \cup \dots \cup \text{var}(t_n).$$

**Definition 4**  $\text{free}$

$\text{free}$  is the total function from the set of formulae to the powerset of  $\text{Var}$  such that:

- i.) For every term  $t_1$ , for every term  $t_2$ ,  

$$\text{free}(t_1 \equiv t_2) = \text{var}(t_1) \cup \text{var}(t_2).$$
- ii.) For every term  $t_1, \dots$ , for every term  $t_n$ , for every  $n$ -ary relation symbol  $R$ ,  $R \in \text{Rel}$ ,  

$$\text{free}(R(t_1, \dots, t_n)) = \text{var}(t_1) \cup \dots \cup \text{var}(t_n).$$
- iii.) For every formula  $\phi$ ,  $\text{free}(\neg\phi) = \text{free}(\phi)$ .
- iv.) For every formula  $\phi_1$ , for every formula  $\phi_2$ , for  $*$   $\in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ ,  

$$\text{free}((\phi_1 * \phi_2)) = \text{free}(\phi_1) \cup \text{free}(\phi_2).$$
- v.) For every  $v \in \text{Var}$ , for every formula  $\phi$ ,  

$$\text{free}(\forall v \phi) = \text{free}(\phi) \setminus \{v\}.$$
- vi.) For every  $v \in \text{Var}$ , for every formula  $\phi$ ,  

$$\text{free}(\exists v \phi) = \text{free}(\phi) \setminus \{v\}.$$

For each formula  $\phi$ ,  $\text{free}(\phi)$  is the set of variables which occur free in  $\phi$ .

For each formula of the form  $Qv \phi$  (with  $Q$  a quantifier), we call  $\phi$  the *scope* of the quantifier  $Q$ . We say that in a formula  $Qv \phi$  the quantifier  $Q$  binds all instances of the variable  $v$  which occur free in  $\phi$ .

**Semantics** Let  $D$  be a set of objects, called the *domain* of our first order terms and formulae, and  $I$  a total function from  $\text{Const} \cup \text{Func} \cup \text{Rel}$  to  $\bigcup_{n \in \mathbb{N}} D_1 \times \dots \times D_n$

which assigns to each  $c \in \text{Const}$  an element of  $D$ ; to each  $n$ -ary function symbol  $F \in \text{Func}$  a function from  $D^n$  to  $D$ ; and to each  $n$ -ary relation symbol  $R \in \text{Rel}$  a subset of  $D_1 \times \dots \times D_n$ .

Let  $M = \langle D, I \rangle$ . We will call each  $M$  a *model*.

Let  $g$  be a function in  $D^{\text{Var}}$  which assigns to each variable in  $\text{Var}$  an object in the domain  $D$ . We call each  $g$  an *assignment function*.

**Definition 5 Term Interpretation**

Let  $\mathcal{M}$  be a model and  $g$  an assignment function.

- i.) For every  $v \in \text{Var}$ ,  $\llbracket v \rrbracket^{\mathcal{M},g} = g(v)$ .
- ii.) For every  $c \in \text{Const}$ ,  $\llbracket c \rrbracket^{\mathcal{M},g} = \mathsf{l}(c)$ .
- iii.) For every term  $t_1, \dots$ , for every term  $t_n$ , for every  $n$ -ary function symbol  $F$ ,  $F \in \text{Func}$ ,
 
$$\llbracket F(t_1, \dots, t_n) \rrbracket^{\mathcal{M},g} = \mathsf{l}(F) \left( \langle \llbracket t_1 \rrbracket^{\mathcal{M},g}, \dots, \llbracket t_n \rrbracket^{\mathcal{M},g} \rangle \right).$$

Assume that  $d \in \mathsf{D}$  and  $v$  is a variable. In what follows we will use the notation  $g^d$  for the assignment function  $g'$  which differs from the assignment function  $g$  in the following way:

$$\text{For each } x \in \text{Var}, g^d(x) = \begin{cases} d & \text{if } x = v, \text{ and} \\ g(x) & \text{otherwise.} \end{cases}$$

**Definition 6 Formula Validation**

Let  $\mathcal{M} = \langle \mathsf{D}, \mathsf{l} \rangle$  be a model and  $g$  an assignment function.

- i.) For every term  $t_1$ , for every term  $t_2$ ,
 
$$\mathsf{V}^{\mathcal{M},g}(t_1 \equiv t_2) = 1 \text{ iff } \llbracket t_1 \rrbracket^{\mathcal{M},g} = \llbracket t_2 \rrbracket^{\mathcal{M},g}.$$
- ii.) For every term  $t_1, \dots$ , for every term  $t_n$ , for every  $n$ -ary relation symbol  $R$ ,  $R \in \text{Rel}$ ,
 
$$\mathsf{V}^{\mathcal{M},g}(R(t_1, \dots, t_n)) = 1 \text{ iff } \langle \llbracket t_1 \rrbracket^{\mathcal{M},g}, \dots, \llbracket t_n \rrbracket^{\mathcal{M},g} \rangle \in \mathsf{l}(R).$$
- iii.) For every formula  $\phi$ ,
 
$$\mathsf{V}^{\mathcal{M},g}(\neg\phi) = 1 \text{ iff } \mathsf{V}^{\mathcal{M},g}(\phi) = 0.$$
- iv.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,
 
$$\mathsf{V}^{\mathcal{M},g}((\phi_1 \wedge \phi_2)) = 1 \text{ iff } \mathsf{V}^{\mathcal{M},g}(\phi_1) = 1 \text{ and } \mathsf{V}^{\mathcal{M},g}(\phi_2) = 1.$$
- v.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,
 
$$\mathsf{V}^{\mathcal{M},g}((\phi_1 \vee \phi_2)) = 1 \text{ iff } \mathsf{V}^{\mathcal{M},g}(\phi_1) = 1 \text{ or } \mathsf{V}^{\mathcal{M},g}(\phi_2) = 1.$$
- vi.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,
 
$$\mathsf{V}^{\mathcal{M},g}((\phi_1 \rightarrow \phi_2)) = 1 \text{ iff } \mathsf{V}^{\mathcal{M},g}(\phi_1) = 0 \text{ or } \mathsf{V}^{\mathcal{M},g}(\phi_2) = 1.$$
- vii.) For every formula  $\phi_1$ , for every formula  $\phi_2$ ,
 
$$\mathsf{V}^{\mathcal{M},g}((\phi_1 \leftrightarrow \phi_2)) = 1 \text{ iff } \mathsf{V}^{\mathcal{M},g}(\phi_1) = \mathsf{V}^{\mathcal{M},g}(\phi_2).$$
- viii.) For every  $v \in \text{Var}$ , for every formula  $\phi$ ,
 
$$\mathsf{V}^{\mathcal{M},g}(\forall v \phi) = 1 \text{ iff for all } d \in \mathsf{D}, \mathsf{V}^{\mathcal{M},g^d}(\phi) = 1.$$
- ix.) For every  $v \in \text{Var}$ , for every formula  $\phi$ ,
 
$$\mathsf{V}^{\mathcal{M},g}(\exists v \phi) = 1 \text{ iff for at least one } d \in \mathsf{D}, \mathsf{V}^{\mathcal{M},g^d}(\phi) = 1.$$

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