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Constraint-based Computational Semantics

Syntax and Semantics of a Higher Order Logic

Types All expressions of our higher-order language, $\mathcal{L}_{\mathsf{Type}}$, will be typed. To keep the language simple, we will only use two basic types, e (for the basic entities in the domain) and t (for the truth values 0 and 1).

Definition 1 Types
Type *is the smallest set such that*

i.) $e \in \mathsf{Type}$,

ii.) $t \in \mathsf{Type}$,

iii.) *for each* $\tau_1 \in \mathsf{Type}$, *for each* $\tau_2 \in \mathsf{Type}$, $\langle \tau_1, \tau_2 \rangle \in \mathsf{Type}$.

Syntax The basic expressions of $\mathcal{L}_{\mathsf{Type}}$ consist only of variables and constants. In contrast to first order logic there is no distinction between terms and formulae.

Definition 2 Basic Expressions

i.) *For each* $\tau \in \mathsf{Type}$, Var_τ *is the smallest set such that for each* $n \in \mathbb{N}_0$, $v_{n,\tau} \in \mathsf{Var}_\tau$.

ii.) *For each* $\tau \in \mathsf{Type}$, Const_τ *is the smallest set such that for each* $n \in \mathbb{N}_0$, $c_{n,\tau} \in \mathsf{Const}$.

We write Var for the set of all variables, $\bigcup_{\tau \in \mathsf{Type}} \mathsf{Var}_\tau$, and Const for the set of all constants, $\bigcup_{\tau \in \mathsf{Type}} \mathsf{Const}_\tau$.
The set of *basic expressions* of our language is the union of the set of variables and the set of constants.

Definition 3 Meaningful Expressions
The meaningful expressions *of* $\mathcal{L}_{\mathsf{Type}}$ *are the smallest familiy* $(\mathsf{ME}_\tau)_{\tau \in \mathsf{Type}}$ *such that*

i.) *for each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each variable* $v_{n,\tau} \in \mathsf{Var}_\tau$, $v_{n,\tau} \in \mathsf{ME}_\tau$;

ii.) *for each* $\tau \in \mathsf{Type}$, *for each* $n \in \mathbb{N}_0$, *for each constant* $c_{n,\tau} \in \mathsf{Const}_\tau$, $c_{n,\tau} \in \mathsf{ME}_\tau$;

iii.) *for each* $\tau \in \mathsf{Type}$*, for each* $\phi_\tau \in \mathsf{ME}_\tau$*, for each* $\psi_\tau \in \mathsf{ME}_\tau$*,*
$(\phi_\tau \equiv \psi_\tau)_t \in \mathsf{ME}_t$*;*

iv.) *for each* $\phi_{\langle\tau_2,\tau_1\rangle} \in \mathsf{ME}_{\langle\tau_2,\tau_1\rangle}$*, for each* $\psi_{\tau_2} \in \mathsf{ME}_{\tau_2}$*,*
$\left(\phi_{\langle\tau_2,\tau_1\rangle}\,(\psi_{\tau_2})\right)_{\tau_1} \in \mathsf{ME}_{\tau_1}$*;*

v.) *for each* $\tau_1 \in \mathsf{Type}$*, for each* $\tau_2 \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each* $v_{n,\tau_2} \in \mathsf{Var}$*, for each* $\phi_{\tau_1} \in \mathsf{ME}_{\tau_1}$*,*
$(\lambda v_{n,\tau_2}.\phi_{\tau_1})_{\langle\tau_2,\tau_1\rangle} \in \mathsf{ME}_{\langle\tau_2,\tau_1\rangle}$.

Semantics D_e is a set of entities, and $\mathsf{D}_t = \{0,1\}$. For each $\tau_1 \in \mathsf{Type}$, for each $\tau_2 \in \mathsf{Type}$, $\mathsf{D}_{\langle\tau_1,\tau_2\rangle} = \mathsf{D}_{\tau_2}{}^{\mathsf{D}_{\tau_1}}$ (the set of all functions from D_{τ_1} to D_{τ_2}).
Let I be a function assigning a denotation to each non-logical constant, $c_{n,\tau}$, of $\mathcal{L}_{\mathsf{Type}}$ from the set D_τ.
Let $\mathsf{M} = \langle \mathsf{D_e}, \mathsf{I}\rangle$. We will call each M a *model*.
Let g be a function in $\bigcup_{\tau\in\mathsf{Type}} \left(\mathsf{D}_\tau{}^{\mathsf{Var}_\tau}\right)$ which assigns an object (of the appropriate type) in the domain $\bigcup_{\tau\in\mathsf{Type}} \mathsf{D}_\tau$ to each variable in Var. We call each g an *assignment function*.
Assume that v is a variable of type τ and d is an element of D_τ. We will use the notation $g\overset{d}{v}$ for the assignment function g' which differs from the assignment function g in the following way:
For each $\tau \in \mathsf{Type}$, for each $v \in \mathsf{Var}_\tau$, for each $x \in \mathsf{Var}_\tau$, for each $d \in \mathsf{D}_\tau$,
$$g\overset{d}{v}(x) = \begin{cases} d & \text{if } x = v \text{, and} \\ g(x) & \text{otherwise.} \end{cases}$$

Definition 4 Denotation
Let M *be a model and* g *an assignment function.*

i.) *For each* $\tau \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each variable* $v_{n,\tau} \in \mathsf{Var}_\tau$*,*
$[\![v_{n,\tau}]\!]^{\mathsf{M},g} = g\,(v_{n,\tau})$.

ii.) *For each* $\tau \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each constant* $c_{n,\tau} \in \mathsf{Const}_\tau$*,*
$[\![c_{n,\tau}]\!]^{\mathsf{M},g} = \mathsf{I}\,(c_{n,\tau})$.

iii.) *For each* $\tau \in \mathsf{Type}$*, for each* $\phi_\tau \in \mathsf{ME}_\tau$*, for each* $\psi_\tau \in \mathsf{ME}_\tau$*,*
$[\![\, (\phi_\tau \equiv \psi_\tau)_t \,]\!]^{\mathsf{M},g} = 1$ *iff* $[\![\phi_\tau]\!]^{\mathsf{M},g} = [\![\psi_\tau]\!]^{\mathsf{M},g}$.

iv.) *For each* $\phi_{\langle\tau_2,\tau_1\rangle} \in \mathsf{ME}_{\langle\tau_2,\tau_1\rangle}$*, for each* $\psi_{\tau_2} \in \mathsf{ME}_{\tau_2}$*,*
$[\![\, \left(\phi_{\langle\tau_2,\tau_1\rangle}\,(\psi_{\tau_2})\right)_{\tau_1} \,]\!]^{\mathsf{M},g} = [\![\phi_{\langle\tau_2,\tau_1\rangle}]\!]^{\mathsf{M},g} \left([\![\psi_{\tau_2}]\!]^{\mathsf{M},g} \right)$.

v.) *For each* $\tau_1 \in \mathsf{Type}$*, for each* $\tau_2 \in \mathsf{Type}$*, for each* $n \in \mathbb{N}_0$*, for each* $v_{n,\tau_2} \in \mathsf{Var}$*, for each* $\phi_{\tau_1} \in \mathsf{ME}_{\tau_1}$*,*
$[\![\, (\lambda v_{n,\tau_2}.\phi_{\tau_1})_{\langle\tau_2,\tau_1\rangle} \,]\!]^{\mathsf{M},g}$ *is that function* h *from* D_{τ_2} *to* D_{τ_1} *such that for each* $o \in \mathsf{D}_{\tau_2}$*,* $h(o) = [\![\phi_{\tau_1}]\!]^{\mathsf{M},g\overset{o}{v}_{n,\tau_2}}$.

Standard results tell us that the usual logical connectives and quantifiers can be defined using these five clauses.