Unification Grammars and Descriptive Formalisms: Lecture Notes for a Gradute Level Course at Seminar für Sprachwissenschaft Eberhard-Karls-Universität Tübingen Wilhelmstraße 113 72074 Tübingen Germany

Paul John King*

Spring 1992

*Email: king@sns.neuphilologie.uni-tuebingen.de

Lattice Theory

Partial Order

A partial order is a pair $\langle S, \leq \rangle$, where S is a set and $\leq$ is a binary relation on S such that for each s_1, s_2 and $s_3 \in S$,

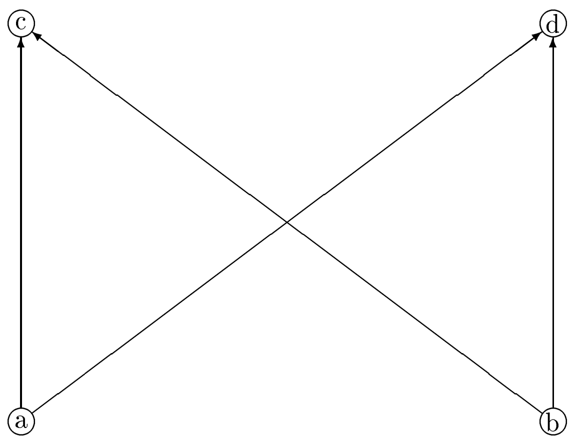
$s_1 \leq s_1$,

if $s_1 \leq s_2$ and $s_2 \leq s_1$ then $s_1 = s_2$, and

if $s_1 \leq s_2$ and $s_2 \leq s_3$ then $s_1 \leq s_3$.

Example 1

If $X = \{a, b, c, d\}$, and $x_1 \sqsubseteq x_2$ iff there is a path (possibly empty) from node x_1 to node x_2 in the graph



then $\langle X, \sqsubseteq \rangle$ is a partial order.

Top

If $\langle S, \leq \rangle$ is a partial order then $s \in S$ is the top of $\langle S, \leq \rangle$ iff for each $s' \in S$, $s' \leq s$.

Bottom

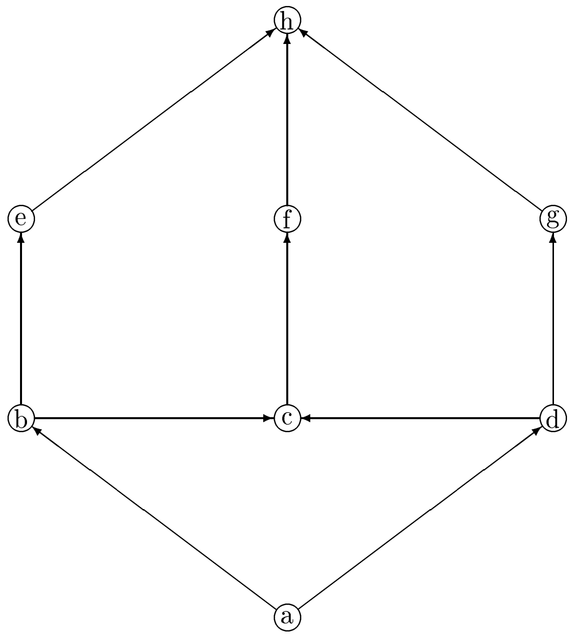
If $\langle S, \leq \rangle$ is a partial order then $s \in S$ is the bottom of $\langle S, \leq \rangle$ iff for each $s' \in S$, $s \leq s'$.

Bounded Partial Order

A bounded partial order is a partial order with a top and bottom.

Example 2

If $X = \{a, b, c, d, e, f, g, h\}$, and $x_1 \sqsubseteq x_2$ iff there is a path (possibly empty) from node x_1 to node x_2 in the graph



then $\langle X, \sqsubseteq \rangle$ is a bounded partial order with top h and bottom a.

Example 3

If X is the set of natural numbers, and $x_1 \sqsubseteq x_2$ iff x_1 divides x_2 then $\langle X, \sqsubseteq \rangle$ is a bounded partial order with top 0 and bottom 1.

Example 4

If S is an arbitrary set then $\langle Pow(S), \subseteq \rangle$ is a bounded partial order with top S and bottom $\emptyset$.

Example 5

For each partial order $\langle S, \leq \rangle$, let

$$\mathit{Close}\langle S, \leq \rangle = \{s \subseteq S \mid \text{if } x_1 \in s \text{ and } x_2 \leq x_1 \text{ then } x_2 \in s\}.$$

For example, if $\langle S, \leq \rangle$ is the partial order in Example 1 then

$$\mathit{Close}\langle S, \leq \rangle = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

If $\langle S, \leq \rangle$ is an arbitrary partial order then $\langle \mathit{Close}\langle S, \leq \rangle, \subseteq \rangle$ is a bounded partial order with top S and bottom $\emptyset$.

Meet

If $\langle S, \leq \rangle$ is a partial order then $s \in S$ is the meet of s_1 and $s_2 \in S$ iff

$s \leq s_1$,

$s \leq s_2$, and

for each $s' \in S$, if $s' \leq s_1$ and $s' \leq s_2$ then $s' \leq s$.

Join

If $\langle S, \leq \rangle$ is a partial order then $s \in S$ is the join of s_1 and $s_2 \in S$ iff

$s_1 \leq s$,

$s_2 \leq s$, and

for each $s' \in S$, if $s_1 \leq s'$ and $s_2 \leq s'$ then $s \leq s'$.

Lattice

A lattice is a partial order $\langle S, \leq \rangle$ such that each s_1 and $s_2 \in S$ have a meet and a join.

Example 1 is not a lattice: c and d have no meet, and a and b have no join.

Bounded Lattice

A bounded lattice is a bounded partial order $\langle S, \leq \rangle$ such that each s_1 and $s_2 \in S$ have a meet and a join.

Example 2 is a bounded lattice, by inspection. Example 3 is a bounded lattice: the meet of x_1 and x_2 is the highest common factor of x_1 and x_2, and the join of x_1 and x_2 is the lowest common multiple of x_1 and x_2. Examples 4 and 5 are bounded lattices: the meet of x_1 and x_2 is $x_1 \cap x_2$, and the join of x_1 and x_2 is $x_1 \cup x_2$.

Distributive Lattice

A distributive lattice is a lattice $\langle S, \leq \rangle$ with meet $\wedge$ and join $\vee$ such that for each s_1, s_2 and $s_3 \in S$,

$$s_1 \wedge (s_2 \vee s_3) = (s_1 \wedge s_2) \vee (s_1 \wedge s_3), \text{ and}$$
$$s_1 \vee (s_2 \wedge s_3) = (s_1 \vee s_2) \wedge (s_1 \vee s_3).$$

Example 2 is not a distributive lattice: $f \wedge (e \vee g) = f$, whereas $(f \wedge e) \vee (f \wedge g) = c$. By number thoery, example 3 is a distributive lattice. By set theory, examples 4 and 5 are distributive lattices.

Complement

If $\langle S, \leq \rangle$ is a bounded lattice with meet $\wedge$, join $\vee$, top $\top$ and bottom $\bot$ then $s' \in S$ is a complement of $s \in S$ iff

$$s \wedge s' = \bot, \text{ and}$$
$$s \vee s' = \top.$$

Boolean Algebra

A Boolean algebra is a bounded and distributive lattice $\langle S, \leq \rangle$ such that each $s \in S$ has a complement. Each $s \in S$ has a unique complement, since if s_1 and $s_2 \in S$ are both complements of $s \in S$ then

$$\begin{aligned}
s_1 &= s_1 \wedge \top \\
&= s_1 \wedge (s \vee s_2) \\
&= (s_1 \wedge s) \vee (s_1 \wedge s_2) && \langle S, \leq \rangle \text{ is a distributive lattice} \\
&= \bot \vee (s_1 \wedge s_2) \\
&= \bot \vee (s_2 \wedge s_1) \\
&= (s_2 \wedge s) \vee (s_2 \wedge s_1) \\
&= s_2 \wedge (s \vee s_1) && \langle S, \leq \rangle \text{ is a distributive lattice} \\
&= s_2 \wedge \top \\
&= s_2.
\end{aligned}$$

Example 3 is not a Boolean algebra: there is no $x \in X$ such that the highest common factor of 2 and x is 1 and the lowest common multiple of 2 and x is 0. Examples 4 is a Boolean algebra: for each $x \in \mathit{Pow}(S)$, $x \cap (S \setminus x) = \emptyset$, and $x \cup (S \setminus x) = S$. Example 5 is sometimes a Boolean algebra, and sometimes not, depending on the partial order that generates the example. For example, if $S = \{0\}$ and $0 \leq 0$ then $\langle \mathit{Close}\langle S, \leq \rangle, \subseteq \rangle$ is a Boolean algebra, but if $\langle S, \leq \rangle$ is the partial order in Example 1 then $\langle \mathit{Close}\langle S, \leq \rangle, \subseteq \rangle$ is not a Boolean algebra. However, Example 5 is *almost* a Boolean algebra.

Relative Pseudo Complement

If $\langle S, \leq \rangle$ is a lattice with meet $\wedge$ then $s \in S$ is the relative pseudocomplement of s_1 and $s_2 \in S$ iff

$s_1 \wedge s \leq s_2$, and

for each $s' \in S$, if $s_1 \wedge s' \leq s_2$ then $s' \leq s$.

Heyting Algebra

A Heyting algebra is a bounded and distributive lattice $\langle S, \leq \rangle$ such that each s_1 and $s_2 \in S$ has a relative pseudo complememnt.

Every Boolean algebra is a Heyting algebra: if $\langle S, \leq \rangle$ is a Boolean algebra with complement $\neg$ and join $\vee$ then the relative pseudo complement of s_1 and $s_2 \in S$ is $(\sim s_1) \vee s_2$. Thus, example 4 is a Heyting algebra. Example 5 is a Boolean algebra, regardless of the partial order that generates the example: the relative pseudo complement of s_1 and $s_2 \in \mathit{Close}(S)$ is

$\bigcup\{s \in S \mid s_1 \cap s \subseteq s_2\}$.

For example, if $\langle S, \leq \rangle$ is the partial order in Example 1 then the relative pseudo complement of $\{a\}$ and $\{b\} \in \mathit{Close}\langle S, \leq \rangle$ is $\emptyset$.

Pseudo Complement

If $\langle S, \leq \rangle$ is a Heyting algebra with relative pseudo complement $\rightarrow$ and bottom $\perp$ then the pseudo complement of $s \in S$ is $s \rightarrow \perp$.

In example 5, if $\langle S, \leq \rangle$ is the partial order in Example 1 then the pseudo complement of $\{a\} \in \mathit{Close}\langle S, \leq \rangle$ is $\{b\}$.

Feature Structures

Preliminaries

Let $\mathfrak{A}$ be a fixed set of attributes. Let $\mathfrak{S}$ be the set of finite strings from $\mathfrak{A}$. Let ε be the empty string. Let $\langle \mathfrak{T}, \preceq \rangle$ be a fixed partial order of types. Read $t_1 \preceq t_2$ as "t_1 is a subtype of t_2".

Basic Feature Structures

A basic feature structure is a pair $\langle g, l \rangle$, where $g \subseteq \mathfrak{S} \times \mathfrak{S}$ and $l : \mathfrak{S} \rightharpoonup \mathfrak{T}$ such that

if $\langle s_1, s_2 \rangle \in g$ then $\langle s_2, s_1 \rangle \in g$,

if $\langle s_1, s_2 \rangle \in g$ and $\langle s_2, s_3 \rangle \in g$ then $\langle s_1, s_3 \rangle \in g$,

if $a \in \mathfrak{A}$, $\langle s_1, s_2 \rangle \in g$ and $\langle s_1 a, s_1 a \rangle \in g$ then $\langle s_1 a, s_2 a \rangle \in g$,

if $a \in \mathfrak{A}$ and $\langle sa, sa \rangle \in g$ then $\langle s, s \rangle \in g$,

$\langle \varepsilon, \varepsilon \rangle \in g$,

$\langle s, s \rangle \in g$ iff $l(s)$ is defined,

if $\langle s_1, s_2 \rangle \in g$ then $l(s_1) = l(s_2)$, and

g is finite.

Let $\mathfrak{B}$ be the set of basic feature structures. Let $\sqsubseteq$ be the binary relation on $\mathfrak{B}$ such that $\langle g_1, l_1 \rangle \sqsubseteq \langle g_2, l_2 \rangle$ iff

$g_2 \subseteq g_1$, and

for each $s \in Dom(l_2)$, $l_1(s) \preceq l_2(s)$.

Read $b_1 \sqsubseteq b_2$ as "b_1 extends b_2" or "b_2 subsumes b_1". $\langle \mathfrak{B}, \sqsubseteq \rangle$ is a partial order.

A Comparison with Pollard and Sag's Basic Feature Structures

Let $\mathfrak{M}$ be the class of connected, acyclic finite state machines that output a type at every state. Let $\mathfrak{C}$ be the set of isomorphism classes of $\mathfrak{M}$. Let $\ll$ be the binary relation on $\mathfrak{C}$ such that $c_1 \ll c_2$ iff there exists a machine homomorhism from some member of c_2 to some member of c_1. Pollard and Sag define a basic feature structure as a member of $\mathfrak{C}$ and read $c_1 \ll c_2$ as "c_1 extends c_2" or "c_2 subsumes c_1".

Let $\gamma : \mathfrak{M} \to Pow(\mathfrak{S} \times \mathfrak{S})$ by $\langle s_1, s_2 \rangle \in \gamma(m)$ iff

m accepts string s_1 and goes to state q_1,

m accepts string s_2 and goes to state q_2, and

$q_1 = q_2$;

let $\lambda : m \to Pow(\mathfrak{S} \times \mathfrak{T})$ such that $\langle s, t\rangle \in \lambda(m)$ iff

m accepts string s and goes to state q, and

m outputs type t in state q;

let $\beta : \mathfrak{M} \to \mathfrak{B}$ by

$\beta(m) = \langle \gamma(m), \lambda(m)\rangle$;

and let $\widehat{\beta} : \mathfrak{C} \to \mathfrak{B}$ such that $\widehat{\beta}(c) = \langle g, l\rangle$ iff

for some $m \in c$, $\beta(m) = \langle g, l\rangle$.

Then

$\widehat{\beta}$ is a bijection, and

$c_1 \ll c_2$ iff $\widehat{\beta}(c_1) \sqsubseteq \widehat{\beta}(c_2)$.

Thus, partial orders $\langle \mathfrak{B}, \sqsubseteq\rangle$ and $\langle \mathfrak{C}, \ll\rangle$ are isomorphic.

Feature Structures

A feature structure is a member of $\mathfrak{F}$, where $\mathfrak{F} = Close\langle \mathfrak{B}, \sqsubseteq\rangle$. Read $f_1 \subseteq f_2$ as "f_2 subsumes f_1". $\langle \mathfrak{F}, \subseteq\rangle$ is a Heyting algebra.

A Comparison with Pollard and Sag's Feature Structures

Let $b_1 \sqsubset b_2$ iff $b_1 \sqsubseteq b_2$ and $b_1 \neq b_2$. Let $\mathfrak{P} = \{p \subseteq \mathfrak{B} \mid$ for no b_1 and $b_2 \in p$, $b_1 \sqsubset b_2\}$. Let $\trianglelefteq$ be the binary relation on $\mathfrak{P}$ such that $p_1 \trianglelefteq p_2$ iff for each $b_1 \in p_1$ there is some $b_2 \in p_2$ such that $b_1 \sqsubseteq b_2$. Pollard and Sag define a feature structure as a member of $\mathfrak{P}$ and read $p_1 \trianglelefteq p_2$ as "p_2 subsumes p_1".

Let $\pi : \mathfrak{F} \to \mathfrak{P}$ by

$\pi(f) = \{b \in f \mid$ for no $b' \in f$, $b \sqsubset b'\}$,

and let $\varphi : \mathfrak{P} \to \mathfrak{F}$ by

$\varphi(p) = \{b \in \mathfrak{B} \mid$ for some $b' \in p$, $b \sqsubseteq b'\}$.

Since there is no set $\{b_i \in \mathfrak{B} \mid i$ is a natural number$\}$ such that $b_i \sqsubset b_{i+1}$ for each natural number i,

$\varphi(\pi(f)) = f$,

$\pi(\varphi(p)) = p$, and

$p_1 \trianglelefteq p_2$ iff $\varphi(p_1) \subseteq \varphi(p_2)$

for each $f \in \mathfrak{F}$ and each p, p_1 and $p_2 \in \mathfrak{P}$. Thus

φ is a bijection, and

$p_1 \trianglelefteq p_2$ iff $\varphi(p_1) \subseteq \varphi(p_2)$.

Thus, Heyting algebras $\langle \mathfrak{F}, \subseteq\rangle$ and $\langle \mathfrak{P}, \trianglelefteq\rangle$ are isomorphic.

Signatures and Structures

Signatures

A **signature** is a function $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ where

$\mathfrak{V}$ is a finite set of **varieties**, and

$\mathfrak{A}$ is a set of **attributes**.

Structures

If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature then an $\mathfrak{S}$-**structure** is a triple $\langle U, V, A\rangle$, where

U is a set of **objects**,

$V : U \to \mathfrak{V}$ is a **variety assignment** function, and

$A : \mathfrak{A} \to \{U \rightharpoonup U\}$ is an **attribute interpretation** function

such that for each $a \in \mathfrak{A}$ and $u \in U$,

S1: if $A(a)$ is defined on u then $V(A(a)(u)) \in \mathfrak{S}\langle V(u), a\rangle$, and

S2: if $\mathfrak{S}\langle V(u), a\rangle \neq \emptyset$ then $A(a)$ is defined on u.

An Explanation

Roughly speaking, each variety and attribute is a meaningless syntactic entity that only gains meaning when it is interpreted in a structure, and the signature encodes a relationship that must hold between varieties and attributes when they are so interpreted. A first intuition is that each object in a structure belongs to a unique category. A structure gives each variety a meaning by using it to name a category. A variety assignment assigns each object the name of the category to which it belongs. A second intuition is that objects in a structure are related by various partial functions. A structure gives each attribute a meaning by using it to name a partial function. An attribute interpretation interprets each attribute as the partial function it names. The signature encodes the following desired relationship between varieties and attributes:

$\mathfrak{S}\langle v, a\rangle = s$ iff the interpretation of attribute a acts upon each object assigned variety v to yield an object assigned some variety in s.

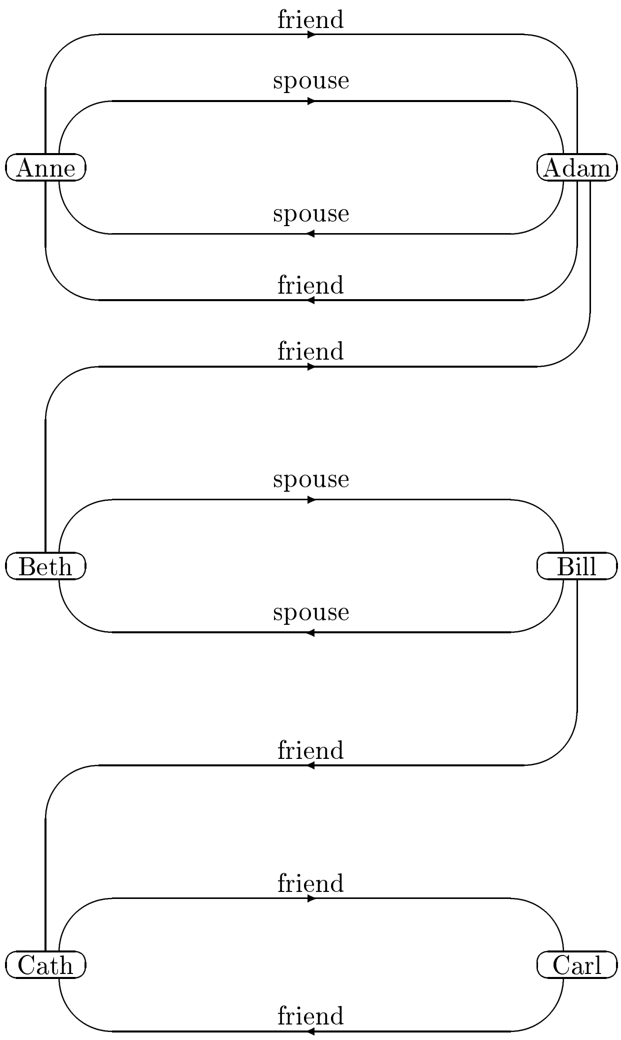
Conditions S1 and S2 enforce this relationship. Condition S1 ensures that if the interpretation of a is defined on an object of variety v then it yields an object of a variety licensed by the signature. Condition S2 ensures that if the signature licenses the interpretation of a to act upon an object of variety v then it must act on every object of variety v.

An Example

I illustrate a signature and a structure with a simple example. Suppose

> Anne and Adam are a married couple, as are Beth and Bill, but Cath and Carl are unmarried. Anne and Adam are best friends, as are Cath and Carl, but Beth's best friend is Adam and Bill's best friend is Cath.

We could picture this scenario as



A signature $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ for this scenario might be as follows. Let

$\mathfrak{V} = \{married, single\}$,

$\mathfrak{A} = \{\textsc{spouse}, \textsc{friend}\}$, and

$\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ by

$\mathfrak{S}\langle married, \textsc{spouse}\rangle = \{married\}$,

$\mathfrak{S}\langle married, \textsc{friend}\rangle = \{married, single\}$,

$\mathfrak{S}\langle single, \textsc{spouse}\rangle = \emptyset$, and

$\mathfrak{S}\langle single, \textsc{friend}\rangle = \{married, single\}$.

A structure $S = \langle U, V, A\rangle$ for this scenario might be as follows. The objects that interest us are the people Anne, Adam, Beth, Bill, Cath and Carl. Let

$U = \{$Anne,Adam,Beth,Bill,Cath,Carl$\}$.

Everyone is either married or not. Let *married* name the category of married people, and let *single* name the category of unmarried people. Let $V : U \to \mathfrak{V}$ by

V(Anne) = *married*,

V(Adam) = *married*,

V(Beth) = *married*,

V(Bill) = *married*,

V(Cath) = *single*, and

V(Carl) = *single*.

Every married person has a unique spouse, and everyone has a unique best friend. Let SPOUSE name the partial function that maps each person to their spouse, and let FRIEND name the total function that maps each person to their best friend. Let $A : \mathfrak{A} \to \{U \rightharpoonup U\}$ by

A(SPOUSE)(Anne) = Adam,

A(SPOUSE)(Adam) = Anne,

A(SPOUSE)(Beth) = Bill,

A(SPOUSE)(Bill) = Beth,

A(FRIEND)(Anne) = Adam,

A(FRIEND)(Adam) = Anne,

A(FRIEND)(Beth) = Adam,

A(FRIEND)(Bill) = Cath,

A(FRIEND)(Cath) = Carl, and

A(FRIEND)(Carl) = Cath.

$\mathfrak{S}$ encodes certain facts about marriage. Firstly, if x is unmarried then

$$\mathfrak{S}\langle V(x), \text{SPOUSE}\rangle = \emptyset.$$

Thus, by condition S1, if x had a spouse y then

$$V(y) \in \emptyset.$$

Thus x has no spouse. $\mathfrak{S}$ encodes that an unmarried person has no spouse. Secondly, if x is married then

$$\mathfrak{S}\langle V(x), \text{SPOUSE}\rangle = \{\mathit{married}\}.$$

Thus, by condition S2, x has a spouse y. Further, by condition S1, y must be married. $\mathfrak{S}$ encodes that a married person has a married spouse.

A Comparison with Pollard and Sag's Type Hierarchy, Attribute Set and Appropriateness Function

Let $\langle \mathfrak{T}, \sqsubseteq\rangle$ be Pollard and Sag's partial order of types, let $\mathfrak{A}$ be their set of attributes, and let $\alpha : \mathfrak{T} \times \mathfrak{A} \rightharpoonup \mathfrak{T}$ be their appropriateness partial function. Write $t_1 \sqsubset t_2$ iff $t_1 \sqsubseteq t_2$ and $t_1 \neq t_2$. Call $t_1 \in \mathfrak{T}$ an immediate subtype of $t_2 \in \mathfrak{T}$ iff $t_1 \sqsubset t_2$ and for no $t \in \mathfrak{T}$, $t_1 \sqsubset t$ and $t \sqsubset t_2$. Call $t \in \mathfrak{T}$ a minimal type iff for no $t' \in \mathfrak{T}$, $t' \sqsubset t$. Let $\mathfrak{M}$ be the set of minimal types. The following all hold.

$\mathfrak{T}$ is finite.

$\langle \mathfrak{T}, \sqsubseteq\rangle$ has a top element $\top$.

Every object is of type $\top$.

If object u is of type t_1 and $t_1 \sqsubseteq t_2$ then u is of type t_2.

If object u is of type t and t has immediate subtypes then u is of exactly one immediate subtype of t.

If $t_1 \sqsubseteq t_2$ and $\alpha\langle t_2, a\rangle$ is defined then $\alpha\langle t_1, a\rangle$ is defined and $\alpha\langle t_1, a\rangle \sqsubseteq \alpha\langle t_2, a\rangle$.

If $\alpha\langle t, a\rangle$ is defined then attribute a acts upon each object of type t to yield an object of type $\alpha\langle t, a\rangle$.

If t is a minimal type and $\alpha\langle t, a\rangle$ is undefined then attribute a acts upon no object of type t.

Construct a signature as follows. Let $\mathfrak{V} = \mathfrak{M}$ and $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to \mathit{Pow}(\mathfrak{V})$ by

$$\mathfrak{S}\langle v, a\rangle = \begin{cases} \{t \in \mathfrak{M} \mid t \sqsubseteq \alpha\langle v, a\rangle\} & \text{if } \alpha\langle v, a\rangle \text{ is defined, and} \\ \emptyset & \text{if } \alpha\langle v, a\rangle \text{ is undefined.} \end{cases}$$

Feature Structure Denotation

Preliminaries

Let $\langle \mathfrak{T}, \preceq \rangle$ be a partial order of types, let $\mathfrak{A}$ be a set of attributes, and let α be an appropriateness function. Let $\langle \mathfrak{B}, \sqsubseteq \rangle$ be the partial order of basic feature structures, and let $\langle \mathfrak{F}, \subseteq \rangle$ be the Heyting algebra of feature structures. Let $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ be the signature generated by $\langle \mathfrak{T}, \preceq \rangle$, $\mathfrak{A}$ and α.

String Interpretation

If $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure then S-string interpretation is the function $[\ ]_S : \mathfrak{A}^\star \to \{U \rightharpoonup U\}$, where for each $s \in \mathfrak{A}^\star$ and $a \in \mathfrak{A}$,

$[\varepsilon]_S =$ the identity function on U, and

$[sa]_S = A(a) \circ [s]_S$.

Basic Feature Structure Denotation

If $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure then S-basic feature structure denotation is the function $[\![\ ]\!]_S : \mathfrak{B} \to Pow(U)$, where

$$
\begin{aligned}
[\![\langle g, l \rangle]\!]_S = \{u \in U \mid\ & \text{for each } \langle s_1, s_2 \rangle \in g, \\
& \qquad [s_1]_S \text{ is defined on u,} \\
& \qquad [s_2]_S \text{ is defined on u, and} \\
& \qquad [s_1]_S(u) = [s_2]_S(u); \\
& \text{and for each } \langle s, t \rangle \in l, \\
& \qquad [s]_S \text{ is defined on u, and} \\
& \qquad V([s]_S(u)) \preceq t\}.
\end{aligned}
$$

Feature Structure Denotation

If $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure then S-feature structure denotation is the function $[\![\ ]\!]_S : \mathfrak{F} \to Pow(U)$, where

$[\![f]\!]_S = \bigcup \{[\![b]\!]_S \mid b \in f\}$.

Soundness and Completeness

Using the formal definitions of feature structures and their denotation, I analyse whether subsumption, unification and generalisation are semantically sound and complete. I have the following desideratum.

Desideratum

For each f_1, f_2 *and* $f \in \mathfrak{F}$,

$f_1 \subseteq f_2$ *iff for each* $\mathfrak{S}$*-structure* S, $[\![f_1]\!]_S \subseteq [\![f_2]\!]_S$,

$f = f_1 \cap f_2$ *iff for each* $\mathfrak{S}$*-structure* S, $[\![f]\!]_S = [\![f_1]\!]_S \cap [\![f_2]\!]_S$, *and*

$f = f_1 \cup f_2$ *iff for each* $\mathfrak{S}$*-structure* S, $[\![f]\!]_S = [\![f_1]\!]_S \cup [\![f_2]\!]_S$.

Subsumption

Lemma

If $b_1 \sqsubseteq b_2$ *then for each* $\mathfrak{S}$*-structure* S, $[\![b_1]\!]_S \subseteq [\![b_2]\!]_S$.

Proposition

If $f_1 \subseteq f_2$ *then for each* $\mathfrak{S}$*-structure* S, $[\![f_1]\!]_S \subseteq [\![f_2]\!]_S$.

Thus, subsumption is sound. However, it is not complete. Suppose

$\mathfrak{V} = \{v\}$,

$\mathfrak{A} = \{a\}$, and

$\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ by

$\mathfrak{S}\langle v, a\rangle = \{v\}$.

Let

$g_1 = \{\langle \varepsilon, \varepsilon\rangle\}$,

$l_1 = \{\langle \varepsilon, v\rangle\}$,

$g_2 = \{\langle \varepsilon, \varepsilon\rangle, \langle a, a\rangle\}$, and

$l_2 = \{\langle \varepsilon, v\rangle, \langle a, v\rangle\}$.

$\langle g_1, l_1\rangle$ and $\langle g_2, l_2\rangle \in \mathfrak{B}$. $\langle g_1, l_1\rangle$ and $\langle g_2, l_2\rangle$ can be pictured as

$\langle g_1, l_1\rangle$ $\langle g_2, l_2\rangle$

Let

$f_1 = \{b \in \mathfrak{B} \mid b \sqsubseteq \langle g_1, l_1\rangle\}$, and

$f_2 = \{b \in \mathfrak{B} \mid b \sqsubseteq \langle g_2, l_2\rangle\}$.

f_1 and $f_2 \in \mathfrak{F}$. For each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$,

$u \in U$

$\Rightarrow V(u) = v$

$\Rightarrow u \in [\![\langle g_1, l_1 \rangle]\!]_S$

$\Rightarrow u \in [\![f_1]\!]_S$,

and

$u \in U$

$\Rightarrow V(u) = v$

$\Rightarrow V(u) = v$ and $A(a)$ is defined on u by S2

$\Rightarrow V(u) = v$, $A(a)$ is defined on u and $V(A(a)(u)) = v$ by S1

$\Rightarrow V(u) = v$, $[a]_S$ is defined on u and $V([a]_S(u)) = v$

$\Rightarrow u \in [\![\langle g_2, l_2 \rangle]\!]_S$

$\Rightarrow u \in [\![f_2]\!]_S$.

Thus, for each $\mathfrak{S}$-structure S,

$[\![f_1]\!]_S = [\![f_2]\!]_S$.

However,

$f_1 \not\sqsubseteq f_2$.

Unification

Unification is not complete, since subsumption is not complete. Neither is unification sound. Suppose

$\mathfrak{V} = \{v_1, v_2, v_3\}$;

$\mathfrak{A} = \{a_1, a_2, a_3\}$; and

$\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ by

$\mathfrak{S}\langle v_1, a_1 \rangle = \{v_2\}$,

$\mathfrak{S}\langle v_1, a_2 \rangle = \{v_3\}$,

$\mathfrak{S}\langle v_1, a_3 \rangle = \emptyset$,

$\mathfrak{S}\langle v_2, a_1 \rangle = \emptyset$,

$\mathfrak{S}\langle v_2, a_2 \rangle = \emptyset$,

$\mathfrak{S}\langle v_2, a_3 \rangle = \{v_3\}$,

$\mathfrak{S}\langle v_3, a_1 \rangle = \emptyset$,

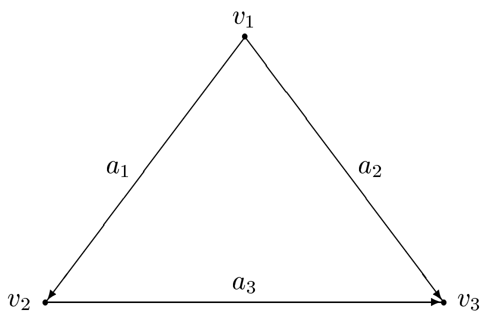
$\mathfrak{S}\langle v_3, a_2 \rangle = \emptyset$, and

$\mathfrak{S}\langle v_3, a_3 \rangle = \{v_2\}$.

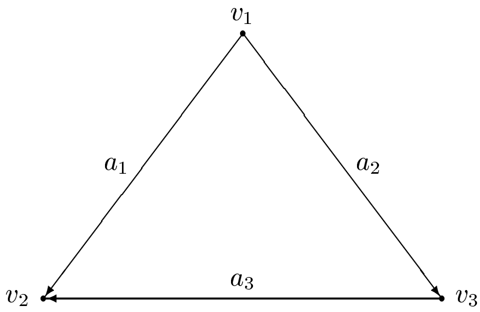
Let

$$\begin{aligned}
g_1 = \{&\langle \varepsilon, \varepsilon \rangle,\\
&\langle \langle a_1 \rangle, \langle a_1 \rangle \rangle,\\
&\langle \langle a_2 \rangle, \langle a_2 \rangle \rangle,\\
&\langle \langle a_1, a_3 \rangle, \langle a_1, a_3 \rangle \rangle,\\
&\langle \langle a_1, a_3 \rangle, \langle a_2 \rangle \rangle,\\
&\langle \langle a_2 \rangle, \langle a_1, a_3 \rangle \rangle \},\\
l_1 = \{&\langle \varepsilon, v_1 \rangle,\\
&\langle \langle a_1 \rangle, v_2 \rangle,\\
&\langle \langle a_2 \rangle, v_3 \rangle,\\
&\langle \langle a_1, a_3 \rangle, v_3 \rangle \},\\
g_2 = \{&\langle \varepsilon, \varepsilon \rangle,\\
&\langle \langle a_1 \rangle, \langle a_1 \rangle \rangle,\\
&\langle \langle a_2 \rangle, \langle a_2 \rangle \rangle,\\
&\langle \langle a_2, a_3 \rangle, \langle a_2, a_3 \rangle \rangle,\\
&\langle \langle a_2, a_3 \rangle, \langle a_1 \rangle \rangle,\\
&\langle \langle a_1 \rangle, \langle a_2, a_3 \rangle \rangle \}, \text{ and}\\
l_2 = \{&\langle \varepsilon, v_1 \rangle,\\
&\langle \langle a_1 \rangle, v_2 \rangle,\\
&\langle \langle a_2 \rangle, v_3 \rangle,\\
&\langle \langle a_2, a_3 \rangle, v_2 \rangle \}.
\end{aligned}$$

$\langle g_1, l_1 \rangle$ and $\langle g_2, l_2 \rangle \in \mathfrak{B}$. $\langle g_1, l_1 \rangle$ and $\langle g_2, l_2 \rangle$ can be pictured as



$\langle g_1, l_1 \rangle$



$\langle g_2, l_2 \rangle$

Let

$f_1 = \{b \in \mathfrak{B} \mid b \sqsubseteq \langle g_1, l_1 \rangle\}$, and

$f_2 = \{b \in \mathfrak{B} \mid b \sqsubseteq \langle g_2, l_2 \rangle\}$.

f_1 and $f_2 \in \mathfrak{F}$. Also

$f_1 \cap f_2 \neq \emptyset$

$\Rightarrow$ for some $\langle g, l \rangle \in \mathfrak{B}$, $\langle g, l \rangle \in f_1$ and $\langle g, l \rangle \in f_2$

$\Rightarrow$ for some $\langle g, l \rangle \in \mathfrak{B}$, $\langle g, l \rangle \sqsubseteq \langle g_1, l_1 \rangle$ and $\langle g, l \rangle \sqsubseteq \langle g_2, l_2 \rangle$

$\Rightarrow$ for some $\langle g, l \rangle \in \mathfrak{B}$, $g_1 \cup g_2 \subseteq g$

$\Rightarrow$ for some $\langle g, l \rangle \in \mathfrak{B}$, $\langle \langle a_2 \rangle, \langle a_1, a_3 \rangle \rangle \in g$ and $\langle \langle a_1 \rangle, \langle a_2, a_3 \rangle \rangle \in g$

$\Rightarrow$ for some $\langle g, l \rangle \in \mathfrak{B}$,

$\langle \langle a_2 \rangle, \langle a_1, a_3 \rangle \rangle \in g$,

$\langle \langle a_1 \rangle, \langle a_2, a_3 \rangle \rangle \in g$,

$\langle \langle a_2, a_3 \rangle, \langle a_1, a_3, a_3 \rangle \rangle \in g$,

$\langle \langle a_1, a_3 \rangle, \langle a_2, a_3, a_3 \rangle \rangle \in g$,

$\langle \langle a_2, a_3, a_3 \rangle, \langle a_1, a_3, a_3, a_3 \rangle \rangle \in g$,

$\langle \langle a_1, a_3, a_3 \rangle, \langle a_2, a_3, a_3, a_3 \rangle \rangle \in g$,

$\ldots$

$\Rightarrow$ for some $\langle g, l \rangle \in \mathfrak{B}$, g is infinite

$\Rightarrow$ a contradiction.

Thus,

$f_1 \cap f_2 = \emptyset$.

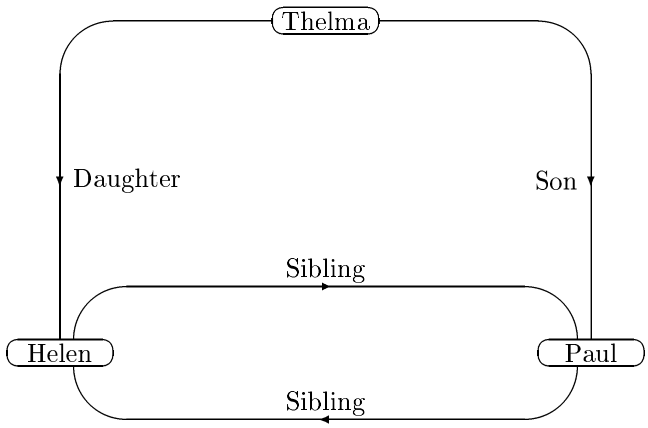
Thus, for each $\mathfrak{S}$-structure S,

$[\![f_1 \cap f_2]\!]_S = \emptyset$.

However, consider the following scenario.

Thelma has one daughter Helen and one son Paul.

We can picture this scenario as follows.



Let

$U = \{\text{Thelma,Helen,Paul}\}$;

$V : U \to \mathfrak{V}$ by

$V(\text{Thelma}) = v_1$,

$V(\text{Helen}) = v_2$, and

$V(\text{Paul}) = v_3$; and

$A : \mathfrak{A} \to \{U \rightharpoonup U\}$ by

$A(a_1) = \text{Daughter}$

$A(a_2) = \text{Son}$

$A(a_3) = \text{Sibling}$.

$S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure. Also,

Thelma $\in [\![f_1]\!]_S \cap [\![f_2]\!]_S$.

Thus, unification is not sound.

Notice that structure S is cyclic: there is a non-empty string of attributes, $\langle a_3, a_3 \rangle$, such that

$[\langle a_3, a_3 \rangle]_S$ is defined on Helen, and

$[\langle a_3, a_3 \rangle]_S(\text{Helen}) = \text{Helen}$.

Does cyclicity make unification unsound?

Call $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$ acyclic iff for each $s \in \mathfrak{A}^\star$ and $u \in U$,

if $[s]_S$ is defined on u and $[s]_S(u) = u$ then $s = \varepsilon$.

Lemma

If

$S = \langle U, V, A\rangle$ *is an acyclic* $\mathfrak{S}$*-structure,*

$\langle g_1, l_1\rangle$ *and* $\langle g_2, l_2\rangle \in \mathfrak{B}$,

$u \in [\![\langle g_1, l_1\rangle]\!]_S$, *and*

$u \in [\![\langle g_2, l_2\rangle]\!]_S$

then, for some $\langle g, l\rangle \in \mathfrak{B}$,

$\langle g, l\rangle \sqsubseteq \langle g_1, l_1\rangle$, $\langle g, l\rangle \sqsubseteq \langle g_2, l_2\rangle$, *and* $u \in [\![\langle g, l\rangle]\!]_S$.

Proof

This is long, technical proof! Feel free to ignore it!! Let

$\mathit{Start} = \mathit{Dom}(g_1) \cup \mathit{Dom}(g_2)$.

Then $[s]_S$ is defined on u for each $s \in \mathfrak{A}^\star$. Ideally, we would let

$g = \{\langle s_1, s_2\rangle \in \mathit{Start} \times \mathit{Start} \mid [s_1]_S(u) = [s_2]_S(u)\}$, and

$l = \{\langle s, v\rangle \in \mathit{Start} \times \mathfrak{V} \mid V([s]_S(u)) = v\}$.

Unfortunately, g might not meet the basic feature structure condition

if $a \in \mathfrak{A}$, $\langle s_1, s_2\rangle \in g$ and $\langle s_1 a, s_1 a\rangle \in g$ then $\langle s_1 a, s_2 a\rangle \in g$. **(1)**

We must first close *Start* under (1).

Call s_1 and $s_2 \in \mathfrak{A}^\star$ equivalent iff

$[s_1]_S$ is defined on u,

$[s_2]_S$ is defined on u, and

$[s_1]_S(u) = [s_1]_S(u)$.

Construct sequences of strings from *Start*, in which each string in a sequence extends by exactly one attribute some string equivalent to its predecessor in that sequence: that is, $\langle s_1 a_1, \ldots, s_n a_n\rangle \in \mathit{Seq}$ iff

$s_1 = \varepsilon$,

for each $1 \leq i \leq n$, $a_i \in \mathfrak{A}$,

for each $1 \leq i \leq n$, $s_i a_i \in \mathit{Start}$, and

for each $1 \leq i < n$, $[s_i a_i]_S(u) = [s_{i+1}]_S(u)$.

By induction on $j - i$, for each $\langle s_1 a_1, \ldots, s_n a_n\rangle \in \mathit{Seq}$ and each $1 \leq i \leq j \leq n$,

$[s_i a_i \ldots a_j]_S$ is defined on u, and

$[s_i a_i \ldots a_j]_S(u) = [s_j a_j]_S(u)$. **(2)**

Claim

For each $\langle s_1 a_1, \ldots, s_n a_n \rangle \in$ *Seq*, $n \leq$ *Card*(*Start*).

Proof of Claim

Suppose for some $\langle s_1 a_1, \ldots, s_n a_n \rangle \in$ *Seq*,

$$n > Card(Start).$$

Then, for some $1 \leq i < j \leq n$,

$$s_i a_i = s_j a_j.$$

However,

$$[s_i a_i a_{i+1} \ldots a_j]_S(u) = [s_j a_j]_S(u). \qquad \text{by (2)}$$

Thus,

$$[s_j a_j a_{i+1} \ldots a_j]_S(u) = [s_j a_j]_S(u).$$

Thus,

$$[a_{i+1} \ldots a_j]_S([s_j a_j]_S(u)) = [s_j a_j]_S(u).$$

However, since S *is acyclic,*

$$a_{i+1} \ldots a_j = \varepsilon.$$

This is a contradiction, since $i < j$. $\diamond$

Thus, *Seq* is finite.

Successive strings in a *Seq* sequence 'hop' across any gaps *Start* might have with respect to condition (1). If all the 'hops' in a sequence are put into a single string, and these new strings collected together we have our required closure. Let

$$Stop = \{a_1 \ldots a_n \in \mathfrak{A}^\star \mid \langle s_1 a_1, \ldots, s_n a_n \rangle \in Seq\}$$

Then

> *Stop* is prefix-closed,
>
> $\varepsilon \in$ *Stop*,
>
> *Stop* is finite,
>
> *Start* $\subseteq$ *Stop*, and
>
> for each $s \in$ *Stop*, $[s]_S$ is defined on u.

Also

Claim

If $a \in \mathfrak{A}$, $s_1 a$ *and* $s_2 \in$ *Stop and* $[s_1]_S(u) = [s_2]_S(u)$ *then* $s_2 a \in$ *Stop*.

Proof of Claim

$$s_1 a \text{ and } s_2 \in Stop \text{ and } [s_1]_S(u) = [s_2]_S(u)$$

$\Rightarrow$ *for some* $s'_1, \ldots, s'_m, s, s''_1, \ldots, s''_n \in \mathfrak{A}^\star$
and $a'_1, \ldots, a'_m, a''_1, \ldots, a''_n \in \mathfrak{A}$,
$s_1 = a'_1 \ldots a'_m$,
$s_2 = a''_1 \ldots a''_n$,
$\langle s'_1 a'_1, \ldots, s'_m a'_m, sa \rangle \in Seq$,
$\langle s''_1 a''_1, \ldots, s''_n a''_n \rangle \in Seq$, *and*
$[s_1]_S(u) = [s_2]_S(u)$

$\Rightarrow$ *for some* $s, s''_1, \ldots, s''_n \in \mathfrak{A}^\star$ *and* $a''_1, \ldots, a''_n \in \mathfrak{A}$,
$s_2 = a''_1 \ldots a''_n$,
$sa \in Start$,
$\langle s''_1 a''_1, \ldots, s''_n a''_n \rangle \in Seq$,
$[s_1]_S(u) = [s_2]_S(u)$,
$[s_1]_S(u) = [s]_S(u)$, *and* *by (2)*
$[s_2]_S(u) = [s''_n a''_n]_S(u)$ *by (2)*

$\Rightarrow$ *for some* $s, s''_1, \ldots, s''_n \in \mathfrak{A}^\star$ *and* $a''_1, \ldots, a''_n \in \mathfrak{A}$,
$s_2 = a''_1 \ldots a''_n$,
$sa \in Start$,
$\langle s''_1 a''_1, \ldots, s''_n a''_n \rangle \in Seq$, *and*
$[s''_n a''_n]_S(u) = [s]_S(u)$

$\Rightarrow$ *for some* $s, s''_1, \ldots, s''_n \in \mathfrak{A}^\star$ *and* $a''_1, \ldots, a''_n \in \mathfrak{A}$,
$s_2 = a''_1 \ldots a''_n$ *and* $\langle s''_1 a''_1, \ldots, s''_n a''_n, sa \rangle \in Seq$

$\Rightarrow s_2 a \in Stop$.

$\diamond$

Let

$g = \{\langle s_1, s_2 \rangle \in Stop \times Stop \mid [s_1]_S(u) = [s_2]_S(u)\}$, and
$l = \{\langle s, v \rangle \in Stop \times \mathfrak{V} \mid V([s]_S(u)) = v\}$.

Then

$\langle g, l \rangle \in \mathfrak{B}$,
$\langle g, l \rangle \sqsubseteq \langle g_1, l_1 \rangle$,
$\langle g, l \rangle \sqsubseteq \langle g_2, l_2 \rangle$, and
$u \in [\![\langle g, l \rangle]\!]_S$.

$\square$

Lemma

For each $\mathfrak{S}$*-structure* S, $[\![f_1 \cap f_2]\!]_S \subseteq [\![f_1]\!]_S \cap [\![f_2]\!]_S$.

Proposition

For each acyclic $\mathfrak{S}$*-structure* S, $[\![f_1 \cap f_2]\!]_S = [\![f_1]\!]_S \cap [\![f_2]\!]_S$.

Proof

Let $S = \langle U, V, A \rangle$ be an acyclic $\mathfrak{S}$-structure. We have that

$$[\![f_1 \cap f_2]\!]_S \subseteq [\![f_1]\!]_S \cap [\![f_2]\!]_S.$$

Conversely,

$$\begin{aligned}
& u \in [\![f_1]\!]_S \cap [\![f_2]\!]_S \\
& \Rightarrow u \in [\![f_1]\!]_S \text{ and } u \in [\![f_2]\!]_S \\
& \Rightarrow \text{for some } \langle g_1, l_1 \rangle \in f_1 \text{ and } \langle g_2, l_2 \rangle \in f_2, \\
& \quad\ u \in [\![\langle g_1, l_1 \rangle]\!]_S \text{ and } u \in [\![\langle g_2, l_2 \rangle]\!]_S \\
& \Rightarrow \text{for some } \langle g, l \rangle \in \mathfrak{B}, \langle g_1, l_1 \rangle \in f_1 \text{ and } \langle g_2, l_2 \rangle \in f_2, \\
& \quad\ u \in [\![\langle g, l \rangle]\!]_S, \langle g, l \rangle \sqsubseteq \langle g_1, l_1 \rangle \text{ and } \langle g, l \rangle \sqsubseteq \langle g_2, l_2 \rangle \\
& \Rightarrow \text{for some } \langle g, l \rangle \in \mathfrak{B}, u \in [\![\langle g, l \rangle]\!]_S, \langle g, l \rangle \in f_1 \text{ and } \langle g, l \rangle \in f_2 \\
& \Rightarrow \text{for some } \langle g, l \rangle \in f_1 \cap f_2, u \in [\![\langle g, l \rangle]\!]_S \\
& \Rightarrow u \in [\![f_1 \cap f_2]\!]_S.
\end{aligned}$$

□

Thus, unification is acyclically sound.

Generalisation

Generalisation is not complete, since subsumption is not complete.

Proposition

For each $\mathfrak{S}$*-structure* S, $[\![f_1 \cup f_2]\!]_S = [\![f_1]\!]_S \cup [\![f_2]\!]_S$.

Proof

$$\begin{aligned}
& u \in [\![f_1 \cup f_2]\!]_S \\
& \Leftrightarrow \text{for some } \langle g, l \rangle \in f_1 \cup f_2, u \in [\![\langle g, l \rangle]\!]_S \\
& \Leftrightarrow \text{for some } \langle g, l \rangle \in f_1, u \in [\![\langle g, l \rangle]\!]_S, \text{ or} \\
& \quad\ \text{for some } \langle g, l \rangle \in f_2, u \in [\![\langle g, l \rangle]\!]_S \\
& \Leftrightarrow u \in [\![f_1]\!]_S \text{ or } u \in [\![f_2]\!]_S \\
& \Leftrightarrow u \in [\![f_1]\!]_S \cup [\![f_2]\!]_S
\end{aligned}$$

□

Thus, generalisation is sound.

Implication and Negation

If $\langle S, \leq \rangle$ is a lattice with meet $\wedge$ then $s \in S$ is the relative pseudocomplement of s_1 and $s_2 \in S$ iff

$s_1 \wedge s \leq s_2$, and

for each $s' \in S$, if $s_1 \wedge s' \leq s_2$ then $s' \leq s$.

A Heyting algebra is a bounded and distributive lattice $\langle S, \leq \rangle$ such that each s_1 and $s_2 \in S$ has a relative pseudo complement. If $\langle S, \leq \rangle$ is a Heyting algebra with relative pseudo complement $\rightarrow$ and bottom $\perp$ then the pseudo complement of $s \in S$ is $s \rightarrow \perp$. $\langle \mathfrak{F}, \subseteq \rangle$ is a Heyting algebra with top $\mathfrak{B}$, bottom $\emptyset$, relative pseudo complement $\rightarrow$ and pseudo complement $\neg$, where for each f_1 and $f_2 \in \mathfrak{F}$,

$$f_1 \rightarrow f_2 = \bigcup \{ f \in \mathfrak{F} \mid f_1 \cap f \subseteq f_2 \}.$$

Pollard and Sag write:

> "We introduced feature structures (in the generalised sense) in order to model disjunctive information. As it turns out, they are sufficient for modelling conditional (implicative) information as well. We begin with conditional information; negative information, as we shall see, is a special case of conditional information. It often comes about that we know of a linguistic object X that *if* it satisfies description A, then it must also satisfy description B. Principles of universal grammar are generally of this form. For example, to simplify very slightly, the *Head Feature Principle* says: if sign X is phrasal and Y is the head daughter of X then X and Y share the same value for the path SYNTAX|LOCAL|HEAD. This principle can be expressed as the CONDITIONAL feature structure:
>
> $$_{phrasal-sign}[\,] \Rightarrow$$
> $$\begin{bmatrix} \text{SYN|LOC|HEAD } \boxed{1} \\ \text{DTRS|HEAD-DTR|SYN|LOC|HEAD } \boxed{1} \end{bmatrix}$$
>
> The symbol '$\Rightarrow$' here denotes a certain binary operation on feature structures. In terms of information content, the $\Rightarrow$ operation can be described as follows: if A and B are feature structures, then A $\Rightarrow$ B is the greatest (i.e. least informative) feature structure whose unification with A is subsumed by B." [*Pollard and Sag 1987*, page 43]
>
> "Mathematically speaking, $\Rightarrow$ is the *relative pseudocomplement* operation in the subsumption lattice. Because this operation can be shown to be well defined, the subsumption lattice of feature structures constitutes a distributive lattice of a special kind called a *Heyting algebra*." [*Pollard and Sag*, footnote 12, page 44]

> "The last of the logical notions we consider for feature structures is *negation*. This is just a special case of $\Rightarrow$: if A is a feature structure, then its negation, $\neg$A, is just A $\Rightarrow \bot$." [*Pollard and Sag*, page 45]

These passages can be rephrased as the following desideratum.[1]

Desideratum

For each f_1, f_2 *and* $f \in \mathfrak{F}$,

$f = f_1 \rightarrow f_2$ iff for each $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$,

$[\![f]\!]_S = (U \setminus [\![f_1]\!]_S) \cup [\![f_2]\!]_S$, *and*

$f = \neg f_1$ iff for each $\mathfrak{S}$-structure S, $[\![f]\!]_S = U \setminus [\![f_1]\!]_S$.

If implication is sound then negation is sound, and if implication is complete then negation is complete. Conversely, if negation is unsound then implication is unsound, and if negation is incomplete then implication is incomplete.

Implication and Negation are Incomplete

Proposition

For some signature $\mathfrak{S}$ and some f_1, f_2 and $f \in \mathfrak{F}$,

for each $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$, $[\![f]\!]_S = (U \setminus [\![f_1]\!]_S) \cup [\![f_2]\!]_S$, and

$f \neq f_1 \rightarrow f_2$.

Proof

For some signature $\mathfrak{S}$ and some f_1 and $f_2 \in \mathfrak{F}$,

$f_1 \not\subseteq f_2$, and

for each $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$, $[\![f_1]\!]_S \subseteq [\![f_2]\!]_S$,

since subsumption is incomplete. For each $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$,

$$[\![\mathfrak{B}]\!]_S = U = (U \setminus [\![f_1]\!]_S) \cup [\![f_2]\!]_S.$$

However,

$$f_1 \cap \mathfrak{B} = f_1 \not\subseteq f_2.$$

Thus,

$$\mathfrak{B} \neq f_1 \rightarrow f_2.$$

□

Thus, implication is incomplete. Suppose f_1 and $f_2 \in \mathfrak{F}$ are as defined on page 2. For each $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$,

$$[\![f_2]\!]_S = U.$$

[1]Please note, I use $\rightarrow$ instead of $\Rightarrow$ to avoid confusion in mathematical proofs, where I use $\Rightarrow$ as the mathematical implication symbol.

Thus, for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$,

$$[\![f_2]\!]_S = U \setminus [\![\emptyset]\!]_S.$$

However

$$\emptyset \cap f_1 = \emptyset.$$

Thus,

$$f_1 \subseteq \neg\emptyset.$$

Therefore, since $f_1 \not\subseteq f_2$,

$$f_2 \neq \neg\emptyset,$$

Thus, negation is also incomplete.

Implication and Negation are Unsound

Suppose f_1 and $f_2 \in \mathfrak{F}$ are as defined on pages 3–5. Then

$$f_1 \cap f_2 = \emptyset.$$

Thus,

$$f_2 \subseteq \neg f_1.$$

Suppose S is the $\mathfrak{S}$-structure defined on page 6. Then

$$\text{Thelma} \in [\![f_2]\!]_S,$$

so

$$\text{Thelma} \in [\![\neg f_1]\!]_S.$$

However,

$$\text{Thelma} \in [\![f_1]\!]_S,$$

so

$$\text{Thelma} \notin U \setminus [\![f_1]\!]_S.$$

Therefore,

$$[\![\neg f_1]\!]_S \neq U \setminus [\![f_1]\!]_S.$$

Thus, negation (and hence implication) is unsound. However, S is a cyclic structure. Could implication and negation, like unification, be generally unsound but acyclically sound?

Proposition

If $S = \langle U, V, A \rangle$ is an acyclic structure and f_1 and $f_2 \in \mathfrak{F}$ then

$$[\![f_1 \rightarrow f_2]\!]_S \subseteq (U \setminus [\![f_1]\!]_S) \cup [\![f_2]\!]_S.$$

Proof

$$f_1 \cap (f_1 \rightarrow f_2) \subseteq f_2.$$

Thus,

$$[\![f_1 \cap (f_1 \rightarrow f_2)]\!]_S \subseteq [\![f_2]\!]_S.$$

Thus,

$$[\![f_1]\!]_S \cap [\![f_1 \rightarrow f_2]\!]_S \subseteq [\![f_2]\!]_S.$$

Thus,

$$[\![f_1 \rightarrow f_2]\!]_S \subseteq (U \setminus [\![f_1]\!]_S) \cup [\![f_2]\!]_S.$$

□

Thus, for each f_1, f_2 and $f \in \mathfrak{F}$,

if $f = f_1 \rightarrow f_2$ then for each acyclic $\mathfrak{S}$-structure $S = \langle U, V, A\rangle$,

$$[\![f]\!]_S \subseteq (U \setminus [\![f_1]\!]_S) \cup [\![f_2]\!]_S.$$

However, for implication to be acyclically sound, we need equality, not subset, in the consequent.

Suppose

$\mathfrak{V} = \{v_1, v_2\}$,

$\mathfrak{A} = \{a_1, a_2\}$, and

$\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \rightarrow Pow(Varieties)$ by

$\mathfrak{S}\langle v_1, a_1\rangle = \{v_2\}$

$\mathfrak{S}\langle v_1, a_2\rangle = \{v_2\}$

$\mathfrak{S}\langle v_2, a_1\rangle = \emptyset$

$\mathfrak{S}\langle v_2, a_2\rangle = \emptyset$

Let

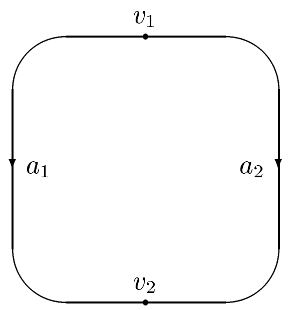
$$\begin{aligned} g_1 = \{&\langle \varepsilon, \varepsilon\rangle, \\ &\langle\langle a_1\rangle, \langle a_1\rangle\rangle, \\ &\langle\langle a_2\rangle, \langle a_2\rangle\rangle, \\ &\langle\langle a_1\rangle, \langle a_2\rangle\rangle, \\ &\langle\langle a_2\rangle, \langle a_1\rangle\rangle\}, \end{aligned}$$

$$\begin{aligned} l_1 = \{&\langle \varepsilon, v_1\rangle, \\ &\langle\langle a_1\rangle, v_2\rangle, \\ &\langle\langle a_2\rangle, v_2\rangle\}, \end{aligned}$$

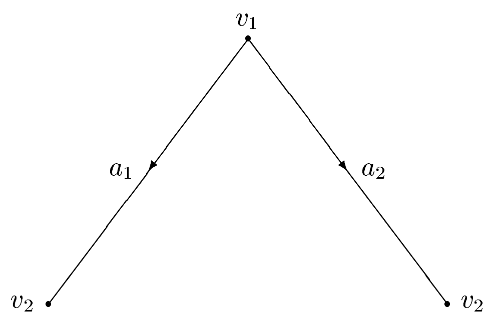
$$\begin{aligned} g_2 = \{&\langle \varepsilon, \varepsilon\rangle, \\ &\langle\langle a_1\rangle, \langle a_1\rangle\rangle, \\ &\langle\langle a_2\rangle, \langle a_2\rangle\rangle\}, \text{ and} \end{aligned}$$

$l_2 = \{\langle \varepsilon, v_1\rangle,$
$\langle\langle a_1\rangle, v_2\rangle,$
$\langle\langle a_2\rangle, v_2\rangle\}.$

$\langle g_1, l_1\rangle$ and $\langle g_2, l_2\rangle \in \mathfrak{B}$. $\langle g_1, l_1\rangle$ and $\langle g_2, l_2\rangle$ can be pictured as



$\langle g_1, l_1\rangle$



$\langle g_2, l_2\rangle$

Let

$f_1 = \{b \in \mathfrak{B} \mid b \sqsubseteq \langle g_1, l_1\rangle\}$, and
$f_2 = \{b \in \mathfrak{B} \mid b \sqsubseteq \langle g_2, l_2\rangle\}$.

f_1 and $f_2 \in \mathfrak{F}$. Let

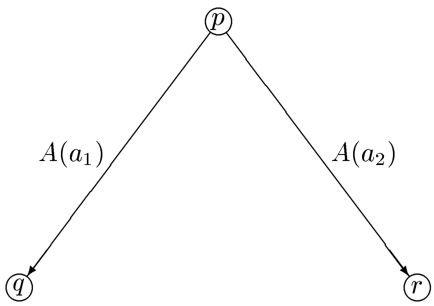
$U = \{p, q, r\}$;
$V : U \to \mathfrak{V}$ by
$V(p) = v_1$,
$V(q) = v_2$, and
$V(r) = v_2$; and

$A : \mathfrak{A} \to \{U \rightharpoonup U\}$ by

$A(a_1) = \{\langle p, q\rangle\}$, and

$A(a_2) = \{\langle p, r\rangle\}$.

$S = \langle U, V, A\rangle$ is an $\mathfrak{S}$-structure. We can picture S as



$p \in [\![f_1]\!]_S$

$\Rightarrow$ for some $\langle g, l\rangle \in f_1$, $p \in [\![\langle g, l\rangle]\!]_S$

$\Rightarrow$ for some $\langle g, l\rangle \sqsubseteq \langle g_1, l_1\rangle$, $p \in [\![\langle g, l\rangle]\!]_S$

$\Rightarrow p \in [\![\langle g_1, l_1\rangle]\!]_S$

$\Rightarrow$ a contradiction,

and

$p \in [\![\neg f_1]\!]_S$

$\Rightarrow$ for some $\langle g, l\rangle \in \neg f_1$, $p \in [\![\langle g, l\rangle]\!]_S$

$\Rightarrow$ for some $\langle g, l\rangle \in \neg f_1$, $\langle g_2, l_2\rangle \sqsubseteq \langle g, l\rangle$

$\Rightarrow \langle g_2, l_2\rangle \in \neg f_1$

$\Rightarrow f_2 \subseteq \neg f_1$

$\Rightarrow f_2 \cap \neg f_1 = \emptyset$

$\Rightarrow$ a contradiction.

Thus,

$p \notin [\![f_1]\!]_S$, and

$p \notin [\![\neg f_1]\!]_S$.

Thus,

$[\![\neg f_1]\!]_S \neq U \setminus [\![f_1]\!]_S$.

Descriptive Formalisms

A description is a syntactic entity that can be interpreted as true or false of an object. The denotation of a description is the collection of precisely those objects of which the interpreted description is true. A descriptive formalism comprises a syntax of descriptions and a denotation function.

Signatures

A **signature** is a function $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ where

$\mathfrak{V}$ is a finite set of **varieties**, and

$\mathfrak{A}$ is a set of **attributes**.

Structures

If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature then an $\mathfrak{S}$-**structure** is a triple $\langle U, V, A\rangle$, where

U is a set of **objects**,

$V : U \to \mathfrak{V}$ is a **variety assignment** function, and

$A : \mathfrak{A} \to \{U \rightharpoonup U\}$ is an **attribute interpretation** function

such that for each $a \in \mathfrak{A}$ and $u \in U$,

S1: if $A(a)$ is defined on u then $V(A(a)(u)) \in \mathfrak{S}\langle V(u), a\rangle$, and

S2: if $\mathfrak{S}\langle V(u), a\rangle \neq \emptyset$ then $A(a)$ is defined on u.

Terms

Let $:$ be a reserved symbol. If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature then an $\mathfrak{S}$-**term** is a member of the smallest set X such that

$: \in X$, and

if $t \in X$ and $a \in \mathfrak{A}$ then $ta \in X$.

Write $[\mathfrak{S}]$ for the set of $\mathfrak{S}$-terms.

Term Interpretations

If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature and $S = \langle U, V, A\rangle$ is an $\mathfrak{S}$-structure then the S-**term interpretation** is the function $[\ ]_S : [\mathfrak{S}] \to \{U \rightharpoonup U\}$, where

$[:]_S$ is the identity function on U, and

if $t \in [\mathfrak{S}]$ and $a \in \mathfrak{A}$ then $[ta]_S = A(a) \circ [t]_S$.

Descriptions

Let (,), $\sim$, $\approx$, $\neg$, $\wedge$, $\vee$ and $\rightarrow$ be reserved symbols. If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature then an $\mathfrak{S}$-**description** is a member of the smallest set X such that

if $t \in [\mathfrak{S}]$ and $v \in \mathfrak{V}$ then $t{\sim}v \in X$,

if t_1 and $t_2 \in [\mathfrak{S}]$ then $t_1{\approx}t_2 \in X$,

if $d \in X$ then $\neg d \in X$,

if d_1 and $d_2 \in X$ then $(d_1{\wedge}d_2) \in X$,

if d_1 and $d_2 \in X$ then $(d_1{\vee}d_2) \in X$, and

if d_1 and $d_2 \in X$ then $(d_1{\rightarrow}d_2) \in X$.

Write $[\![\mathfrak{S}]\!]$ for the set of $\mathfrak{S}$-descriptions.

Description Denotations

If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature and $S = \langle U, V, A\rangle$ is an $\mathfrak{S}$-structure then the S-**denotation** is the function $[\![\ ]\!]_S : [\![\mathfrak{S}]\!] \to Pow(U)$, where

$[\![t{\sim}v]\!]_S = \{u \in U \mid [t]_S$ is defined on u and $V([t]_S(u)) = v\}$,

$[\![t_1{\approx}t_2]\!]_S = \{u \in U \mid [t_1]_S$ is defined on u,
$[t_2]_S$ is defined on u, and
$[t_1]_S(u) = [t_2]_S(u)\}$,

$[\![\neg d]\!]_S = U \setminus [\![d]\!]_S$,

$[\![(d_1{\wedge}d_2)]\!]_S = [\![d_1]\!]_S \cap [\![d_2]\!]_S$,

$[\![(d_1{\vee}d_2)]\!]_S = [\![d_1]\!]_S \cup [\![d_2]\!]_S$, and

$[\![(d_1{\rightarrow}d_2)]\!]_S = (U \setminus [\![d_1]\!]_S) \cup [\![d_2]\!]_S$.

Theories

If $\mathfrak{S}$ is a signature then an $\mathfrak{S}$-**theory** is a subset of $[\![\mathfrak{S}]\!]$.

Theory Denotations

If $\mathfrak{S}$ is a signature and $S = \langle U, V, A\rangle$ is an $\mathfrak{S}$-structure then the S-**denotation** is the function $[\![\ ]\!]_S : Pow[\![\mathfrak{S}]\!] \to Pow(U)$, where

$[\![T]\!]_S = \bigcap\{[\![d]\!]_S \mid d \in T\}$.

Notice that $[\![\emptyset]\!]_S = U$.

Models

If $\mathfrak{S}$ is a signature, $T \subseteq [\![\mathfrak{S}]\!]$ and $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure such that $[\![T]\!]_S = U$ then S is a **model** of T.

A Brief Explanation

Each term is a meaningless string of attributes that only gains meaning when the attributes are interpreted in a structure. A structure gives each attribute a meaning by using it to name a partial function. A term interpretation interprets each term as the composition of the partial functions named by the attributes that make up that term.

Call a description of the form $t{\sim}v$ a varietal description. Loosely speaking, $t{\sim}v$ describes object u iff

> the interpretation of t maps u to some object u', and
>
> the variety of u' is v.

Call a description of the form $t_1{\approx}t_2$ an equational description. Loosely speaking again, $t_1{\approx}t_2$ describes object u iff

> the interpretation of t_1 maps u to some object u_1,
>
> the interpretation of t_2 maps u to some object u_2, and
>
> u_1 and u_2 are the same object.

Call a description of any other form a compound description. Roughly speaking, a description of the form $\neg d$ describes an object iff d does not describe it; a description of the form $(d_1{\wedge}d_2)$ describes an object iff d_1 describes it and d_2 describes it; a description of the form $(d_1{\vee}d_2)$ describes an object iff d_1 describes it or d_2 describes it (or both); a description of the form $(d_1{\rightarrow}d_2)$ describes an object iff whenever d_1 describes it then so does d_2.

A theory is an arbitrary set of descriptions. Like a description, a theory can be interpreted as true or false of an object. Theory T is true of object u iff every description in T is true of u. Additionally, a theory can be interpreted as true or false of a *structure*. Theory T is true of structure S iff T is true of every object in S. We then call S a model of T.

What is it for a linguist to use a descriptive formalism to write a theory of a natural language? Firstly, for a descriptive formalism to be an appropriate tool for the writing of such a theory, the linguist must conceive of the language as a structure, and must believe that an account of the language can be achieved using descriptions alone. Secondly, the linguist must construct a theory that is true of the entire language, that is, a theory that is true of every object in the language. Thus, a successful theory is a theory that has the target natural language as a model.

Metadescriptions

Metadescriptions are not descriptions, but convenient shorthand representations of frequently used descriptions. If $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ is a signature, t, t_1 and $t_2 \in [\mathfrak{S}]$, and d, d_0, $\ldots$, d_n and $d_{n+1} \in [\![\mathfrak{S}]\!]$ then

$t\downarrow$ represents $t{\approx}t$,

$\top$ represents $:\downarrow$,

$\bot$ represents $\neg\top$,

$t_1{\not\approx}t_2$ represents $(\neg t_1{\approx}t_2{\wedge}(t_1{\downarrow}{\wedge}t_2{\downarrow}))$,

$\bigwedge\emptyset$ represents $\top$,

$\bigwedge\{d\}$ represents d,

$\bigwedge\{d_0, \ldots, d_n, d_{n+1}\}$ represents $(\bigwedge\{d_0, \ldots, d_n\}{\wedge}d_{n+1})$,

$\bigvee\emptyset$ represents $\bot$,

$\bigvee\{d\}$ represents d, and

$\bigvee\{d_0, \ldots, d_n, d_{n+1}\}$ represents $(\bigvee\{d_0, \ldots, d_n\}{\vee}d_{n+1})$.

Further, if $\mathfrak{V}$ is generated from a type partial order $\langle \mathfrak{T}, \preceq \rangle$ and $\tau \in \mathfrak{T}$ then

$t{\sim}\tau$ represents $\bigvee\{t{\sim}v \mid v \in \mathfrak{V} \text{ and } v \preceq \tau\}$.

Thus, if $S = \langle U, V, A\rangle$ is an $\mathfrak{S}$-structure then

$[\![t{\downarrow}]\!]_S = \{u \in U \mid [t]_S \text{ is defined on } u\}$,

$[\![\top]\!]_S = U$,

$[\![\bot]\!]_S = \emptyset$,

$$[\![t_1{\not\approx}t_2]\!]_S = \{u \in U \mid [t_1]_S \text{ is defined on } u,\ [t_2]_S \text{ is defined on } u, \text{ and } [t_1]_S(u) \neq [t_2]_S(u)\},$$

$[\![\bigwedge\{d_0, \ldots, d_n\}]\!]_S = \bigcap\{[\![d_0]\!]_S, \ldots, [\![d_n]\!]_S\}$,

$[\![\bigvee\{d_0, \ldots, d_n\}]\!]_S = \bigcup\{[\![d_0]\!]_S, \ldots, [\![d_n]\!]_S\}$, and

$[\![t{\sim}\tau]\!]_S = \{u \in U \mid [t]_S \text{ is defined on } u \text{ and } V([t]_S(u)) \preceq \tau\}$.

Universal Principles (Part 1)

The Head Feature Principle

Informally:

> "if a phrase has a head daughter, then they share the same head features" [*Pollard and Sag 1987*, page 58]

As an AVM:

$$\left[\text{DTRS}\ \ {}_{\textit{headed-structure}}[\ \ ]\right]$$
$$\Rightarrow \begin{bmatrix} \text{SYN}\,|\,\text{LOC}\,|\,\text{HEAD}\ \boxed{1} \\ \text{DTRS}\,|\,\text{H-DTR}\,|\,\text{SYN}\,|\,\text{LOC}\,|\,\text{HEAD}\ \boxed{1} \end{bmatrix}$$

As a description:

> ((:∼*phrasal-sign*∧:DTRS∼*headed-structure*)
>
> →:SYN LOC HEAD≈:DTRS H-DTR SYN LOC HEAD)

The Subcategorisation Principle

Informally:

> "It should be intuitively clear that the SUBCAT value of a phrase ought to be the SUBCAT value of the lexical head minus those specifications that have already been satisfied by some constituent in the phrase." [*Pollard and Sag 1987*, page 71]

As an AVM

$$\left[\text{DTRS}\ \ {}_{\textit{headed-structure}}[\ \ ]\right]$$
$$\Rightarrow \begin{bmatrix} \text{SYN}\,|\,\text{LOC}\,|\,\text{SUBCAT}\ \boxed{2} \\ \text{DTRS}\ \begin{bmatrix} \text{H-DTR}\,|\,\text{SYN}\,|\,\text{LOC}\,|\,\text{SUBCAT append}(\boxed{1}, \boxed{2}) \\ \text{C-DTRS}\ \boxed{1} \end{bmatrix} \end{bmatrix}$$

As a description (almost!):

> ((:∼*phrasal-sign*∧:DTRS∼*headed-structure*)
>
> →:DTRS H-DTR SYN LOC SUBCAT
> ≈append(:DTRS C-DTRS, :SYN LOC SUBCAT))

Lists of Signs

Introduce the type *los* for objects that are lists of signs. Each list of signs is either empty or non empty. Introduce subtypes *elos* and *nelos* of *los*. Empty lists of signs are of type *elos*, and non-empty lists of signs are of type *nelos*. *elos* and *nelos* have no subtypes, so they are both varieties.

No attributes are appropriate for an empty list of signs: for each $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle elos, a\rangle = \emptyset.$$

Only attributes FIRST and REST are appropriate for *nelos*. The FIRST of a non empty list of signs is the first sign on the list. The REST of a non empty list of signs is the list remaining when the first sign is removed. Thus,

$$\mathfrak{S}\langle nelos, \mathrm{FIRST}\rangle = \{lexical\text{-}sign, phrasal\text{-}sign\},$$

$$\mathfrak{S}\langle nelos, \mathrm{REST}\rangle = \{elos, nelos\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle nelos, a\rangle = \emptyset.$$

Append of Signs

If there were attributes LEFT and RIGHT such that the LEFT of list of signs s is list of signs s_l, the right of s is list of signs s_r, and s is the concatenation of s_l and s_r, then we could incorporate concatenation into our formalism immediately. Unfortunately, we cannot interpret LEFT and RIGHT as functions. Does

$$\mathrm{LEFT}\langle s_1, s_2, s_3\rangle = \langle s_1\rangle \text{ and } \mathrm{RIGHT}\langle s_1, s_2, s_3\rangle = \langle s_2, s_3\rangle,$$

or does

$$\mathrm{LEFT}\langle s_1, s_2, s_3\rangle = \langle s_1, s_2\rangle \text{ and } \mathrm{RIGHT}\langle s_1, s_2, s_3\rangle = \langle s_3\rangle?$$

We create a new kind of object called an append of signs. An append of signs primarily comprsises three lists of signs. Our aim is to write a theory that ensures that the first of these lists is the concatenation of the second and the third.

Introduce the type *aos* for objects that are appends of signs. Each append of signs is either finished or unfinished. Introduce subtypes *faos* and *uaos* of *aos*. Finished appends of signs are of type *faos*, and unfinished appends of signs are of type *uaos*. *faos* and *uaos* have no subtypes, so they are both varieties.

Only attributes RES, LEFT and RIGHT are appropriate for *faos*. The RES, LEFT and RIGHT of a finished append of signs are all lists of signs, but the LEFT of a finished append of signs is always empty. Thus,

$$\mathfrak{S}\langle faos, \mathrm{RES}\rangle = \{elos, nelos\},$$

$$\mathfrak{S}\langle faos, \mathrm{LEFT}\rangle = \{elos\},$$

$$\mathfrak{S}\langle faos, \mathrm{RIGHT}\rangle = \{elos, nelos\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle faos, a\rangle = \emptyset.$$

Only attributes RES, LEFT, RIGHT and SUB are appropriate for *uaos*. The RES, LEFT and RIGHT of an unfinished append of signs are all lists of signs, but the LEFT of an unfinished append of signs is always non empty. The SUB of an unfinished append of signs is an append of signs. Thus,

$$\mathfrak{S}\langle uaos, \text{RES}\rangle = \{elos, nelos\},$$

$$\mathfrak{S}\langle uaos, \text{LEFT}\rangle = \{nelos\},$$

$$\mathfrak{S}\langle uaos, \text{RIGHT}\rangle = \{elos, nelos\},$$

$$\mathfrak{S}\langle uaos, \text{SUB}\rangle = \{faos, uaos\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle uaos, a\rangle = \emptyset.$$

Stealing shamelessly from Alex Franz! let

$$d_f = (:\sim\!faos \rightarrow :\text{RES}\approx:\text{RIGHT}), \text{ and}$$

$$\begin{aligned} d_u = (:\sim\!uaos \rightarrow (&:\text{RES FIRST}\approx:\text{LEFT FIRST}\\ &\wedge:\text{SUB RES}\approx:\text{RES REST}\\ &\wedge:\text{SUB LEFT}\approx:\text{LEFT REST}\\ &\wedge:\text{SUB RIGHT}\approx:\text{RIGHT})). \end{aligned}$$

Suppose that structure S is a model of $\{d_u, d_f\}$ and that u is an append of signs in S. Then u's RES is the concatenation of u's LEFT and u's RIGHT.

The Subcategorisation Principle (again!)

Let the subcategorisation of a sign be an append of signs, and let the list of signs previously considered to be the subcategorisation of a sign now be the RES of the subcategorisation of that sign. Then the RES of the subcategorisation of a sign is the concatenation of the LEFT and RIGHT of the subcategorisation, objects that can be shared with other signs, etc. Thus, as a description, the Subcategorisation Principle reads

$$\begin{aligned} &((:\sim\!\textit{phrasal-sign}\wedge:\text{DTRS}\sim\!\textit{headed-structure})\\ &\ \rightarrow(:\text{DTRS H-DTR SYN LOC SUBCAT LEFT}\approx:\text{DTRS C-DTRS}\\ &\quad\wedge:\text{DTRS H-DTR SYN LOC SUBCAT RIGHT}\\ &\quad\ \approx:\text{SYN LOC SUBCAT RES})) \end{aligned}$$

Universal Principles (Part 2)

Preliminaries

Here is a relevent fragment of [*Pollard and Sag 1987*]'s type hierarchy. I use the following notation. Let $\top$ be the top type in the type hierarchy, and let

$$t \prec t' \quad \begin{bmatrix} a_1 & t_1 \\ \vdots & \vdots \\ a_n & t_n \end{bmatrix}$$

mean that t' is the immediate supertype of t and that attributes $a_1, \ldots, a_n$ are appropriate for t, yielding values of type $t_1, \ldots, t_n$ respectively.

$$\textit{semantics} \prec \top \quad \begin{bmatrix} \text{CONT} & \textit{content} \\ \text{INDS} & \textit{set-of-indices} \end{bmatrix}$$

$$\textit{content} \prec \top$$

$$\textit{circumstance} \prec \textit{content}$$

$$\textit{basic-circumstance} \prec \textit{circumstance} \quad \begin{bmatrix} \text{REL} & \top \\ \text{POL} & \top \end{bmatrix}$$

$$\textit{compound-circumstance} \prec \textit{circumstance} \quad \begin{bmatrix} \text{CONN} & \top \\ \text{JUNCTS} & \textit{set-of-circumstances} \end{bmatrix}$$

$$\textit{quantified-circumstance} \prec \textit{circumstance} \quad \begin{bmatrix} \text{QUANT} & \textit{quantifier} \\ \text{SCOPE} & \textit{circumstance} \end{bmatrix}$$

$$\textit{indexed-object} \prec \textit{content} \quad \begin{bmatrix} \text{IND} & \textit{index} \end{bmatrix}$$

$$\textit{referential-object} \prec \textit{indexed-object}$$

$$\textit{quantifier} \prec \textit{indexed-object} \quad \begin{bmatrix} \text{DET} & \top \end{bmatrix}$$

$$\textit{index} \prec \top \quad \begin{bmatrix} \text{VAR} & \top \\ \text{RESTRICT} & \textit{circumstance} \end{bmatrix}$$

$$\textit{constituent-structure} \prec \top$$

$$\textit{headed-structure} \prec \textit{constituent-structure} \quad \begin{bmatrix} \text{H-DTR} & \textit{sign} \\ \text{C-DTRS} & \textit{los} \end{bmatrix}$$

$$\textit{head-filler-structure} \prec \textit{headed-structure}$$
$$\begin{bmatrix} \text{C-DTRS} & \textit{elos} \\ \text{F-DTR} & \textit{sign} \end{bmatrix}$$

$$\textit{head-complement-structure} \prec \textit{headed-structure}$$

Let Σ be the set of minimal subtypes of *content*.

The Semantics Principle

The Semantics Principle reads

$$\begin{bmatrix} \text{DTRS} & {}_{\textit{headed-structure}}[\,] \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \text{SEM} \begin{bmatrix} \text{CONT successively-combine-semantics}(\boxed{1}, \boxed{2}) \\ \text{INDS collect-indices}(\boxed{3}) \end{bmatrix} \\ \text{DTRS } \boxed{3} \begin{bmatrix} \text{H-DTR} \mid \text{SEM} \mid \text{CONT } \boxed{1} \\ \text{C-DTRS } \boxed{2} \end{bmatrix} \end{bmatrix},$$

where, if A and B are contents and L is a list of signs then

combine-semantics(A, B) =

if A has type *circumstance* and B has type *quantifier*

then $\begin{bmatrix} \text{QUANT} & B \\ \text{SCOPE} & A \end{bmatrix}$

else A,

and

successively-combine-semantics(A, L) =

if length$(L) = 0$

then A

else successively-combine-semantics
(combine-semantics$(A, \text{SEM} \mid \text{CONT}$ of first(L)),rest(L)),

and where

> "collect-indices is just the function defined on feature structures of type *constituent-structure* which takes the set union of the INDS values on all the daughters." [*Pollard and Sag 1987*, Page 104]

We split the Semantics Principle into two parts. Part 1 reads

$$\begin{bmatrix} \text{DTRS} & {}_{\textit{headed-structure}}[\,] \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \text{SEM} \mid \text{CONT successively-combine-semantics}(\boxed{1}, \boxed{2}) \\ \text{DTRS} \begin{bmatrix} \text{H-DTR} \mid \text{SEM} \mid \text{CONT } \boxed{1} \\ \text{C-DTRS } \boxed{2} \end{bmatrix} \end{bmatrix},$$

and part 2 reads

$$\left[\text{DTRS} \quad {}_{\textit{headed-structure}}[\;]\right]$$
$$\Rightarrow \begin{bmatrix} \text{SEM} \,|\, \text{INDS collect-indices}(\boxed{3}) \\ \text{DTRS } \boxed{3} \end{bmatrix}.$$

Successively Combined Semantics

As an 'almost' description, part 1 of the Semantics Principle reads

$$((\text{:}{\sim}\textit{phrasal-sign}\wedge\text{:DTRS}{\sim}\textit{headed-structure})$$
$$\rightarrow\text{:SEM CONT}\approx\text{successively-combine-semantics}$$
$$(\text{:DTRS H-DTR SEM CONT, :DTRS C-DTRS}))$$

Unfortunately, the function successively-combine-semantics is not part of our descriptive formalism. However, we can use techniques like those used to append lists of signs in order to successively combine semantics. Again, we create a new kind of object out of existing objects and write a theory that constrains the components of these new objects in the desired way.

Let our new objects be called successively combined semantics. Roughly, a successively combined semantics comprises two contents and a list of signs. Our theory will ensure that, in any model of that theory, the first content is the successively-combine-semantics of the second content and the list of signs.

Introduce the type *scs* for objects that are successively combined semantics. Each successively combined semantics is either finished or unfinished. Introduce subtypes *fscs* and *uscs* of *scs*. Finished successively combined semantics are of type *fscs* and unfinished successively combined semantics are of type *uscs*. *fscs* and *uscs* have no subtypes, so they are both varieties.

Only attributes RES, ARG and LIST are appropriate for *fscs*. The RES and ARG of a finished sucessively combined semantics are contents, and the LIST of a finished successively combined semantics is an empty list of signs. Thus,

$$\mathfrak{S}\langle \textit{fscs}, \text{RES}\rangle = \Sigma,$$

$$\mathfrak{S}\langle \textit{fscs}, \text{ARG}\rangle = \Sigma,$$

$$\mathfrak{S}\langle \textit{fscs}, \text{LIST}\rangle = \{\textit{elos}\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle \textit{fscs}, a\rangle = \emptyset.$$

Only attributes RES, ARG, LIST and SUB are appropriate for *uscs*. The RES and ARG of an unfinished sucessively combined semantics are contents, the LIST of an unfinished successively combined semantics is a non empty list of signs, and the SUB of an unfinished successively combined semantics is a successively combined semantics. Thus,

$$\mathfrak{S}\langle uscs, \text{RES}\rangle = \Sigma,$$
$$\mathfrak{S}\langle uscs, \text{ARG}\rangle = \Sigma,$$
$$\mathfrak{S}\langle uscs, \text{LIST}\rangle = \{nelos\},$$
$$\mathfrak{S}\langle uscs, \text{SUB}\rangle = \{fscs, uscs\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle uscs, a\rangle = \emptyset.$$

Let

$$\begin{aligned}
d_1 &= (:\sim fscs \rightarrow :\text{RES}\approx:\text{ARG}),\\
d_2 &= ((:\sim uscs\\
&\qquad\quad \wedge\neg(:\text{ARG}\sim circumstance\\
&\qquad\qquad\quad \wedge:\text{LIST FIRST SEM CONT RES}\sim quantifier))\\
&\qquad \rightarrow(\text{SUB RES}\approx:\text{RES}\\
&\qquad\quad \wedge:\text{SUB ARG}\approx:\text{ARG}\\
&\qquad\quad \wedge:\text{SUB LIST}\approx:\text{LIST REST})), \text{ and}\\
d_3 &= ((:\sim uscs\\
&\qquad\quad \wedge(:\text{ARG}\sim circumstance\\
&\qquad\qquad \wedge:\text{LIST FIRST SEM CONT RES}\sim quantifier))\\
&\qquad \rightarrow(\text{SUB RES}\approx:\text{RES}\\
&\qquad\quad \wedge:\text{SUB ARG}\sim quantified\text{-}circumstance\\
&\qquad\quad \wedge:\text{SUB ARG QUANT}\approx:\text{LIST FIRST SEM CONT RES}\\
&\qquad\quad \wedge:\text{SUB ARG SCOPE}\approx:\text{ARG}\\
&\qquad\quad \wedge:\text{SUB LIST}\approx:\text{LIST REST})).
\end{aligned}$$

These descriptions are easier to digest as AVMs. We might write

$$d_1 = fscs\,[\,] \rightarrow \begin{bmatrix}\text{RES}\ \boxed{1}\\ \text{ARG}\ \boxed{1}\end{bmatrix},$$

$$d_2 = \left(uscs\,[\,] \wedge\neg\begin{bmatrix}\text{ARG}\ circumstance\,[\,]\\ \text{LIST}\,|\,\text{FIRST}\,|\,\text{SEM}\,|\,\text{CONT}\,|\,\text{RES}\ quantifier\,[\,]\end{bmatrix}\right)$$

$$\rightarrow \begin{bmatrix}\text{RES}\ \boxed{1}\\ \text{ARG}\ \boxed{2}\\ \text{LIST}\,|\,\text{REST}\ \boxed{3}\\ \text{SUB}\ \begin{bmatrix}\text{RES}\ \boxed{1}\\ \text{ARG}\ \boxed{2}\\ \text{LIST}\ \boxed{3}\end{bmatrix}\end{bmatrix}, \text{ and}$$

$$d_3 = \left(uscs[\,] \wedge \begin{bmatrix} \text{ARG } \textit{circumstance}[\,] \\ \text{LIST}\,|\,\text{FIRST}\,|\,\text{SEM}\,|\,\text{CONT}\,|\,\text{RES } \textit{quantifier}[\,] \end{bmatrix} \right)$$

$$\rightarrow \begin{bmatrix} \text{RES } \boxed{1} \\ \text{ARG } \boxed{3} \\ \text{LIST} \begin{bmatrix} \text{FIRST}\,|\,\text{SEM}\,|\,\text{CONT}\,|\,\text{RES } \boxed{2} \\ \text{REST } \boxed{4} \end{bmatrix} \\ \text{SUB} \begin{bmatrix} \text{RES } \boxed{1} \\ \text{ARG } \textit{quantified-circumstance} \begin{bmatrix} \text{QUANT } \boxed{2} \\ \text{SCOPE } \boxed{3} \end{bmatrix} \\ \text{LIST } \boxed{4} \end{bmatrix} \end{bmatrix}.$$

Suppose that structure S is a model of $\{d_1, d_2, d_3\}$ and that u is a successively combined semantics. Then u's RES is the successively-combine-semantics of u's ARG and u's LIST.

The Semantics Principle (Part 1)

Let the SEM | CONT of a sign be a successively combined semantics, and let the content previously considered to be the SEM | CONT of a sign now be the SEM | CONT | RES of that sign. Then the SEM | CONT | RES of a sign is the successively-combine-semantics of the sign's SEM | CONT | ARG and SEM | CONT | LIST, objects that can be shared with other signs, etc. Thus, part 1 of the Semantics Principle reads, as a description

$$((:\sim\textit{phrasal-sign}\wedge:\text{DTRS}\sim\textit{headed-structure})$$
$$\rightarrow(:\text{SEM CONT ARG}\approx:\text{DTRS H-DTR SEM CONT RES}$$
$$\wedge:\text{SEM CONT LIST}\approx:\text{DTRS C-DTRS})),$$

and as an AVM

$$\textit{phrasal-sign}\,[\text{DTRS } \textit{headed-structure}[\,]]$$
$$\rightarrow \begin{bmatrix} \text{SEM}\,|\,\text{CONT} \begin{bmatrix} \text{ARG } \boxed{1} \\ \text{LIST } \boxed{2} \end{bmatrix} \\ \text{DTRS} \begin{bmatrix} \text{H-DTR}\,|\,\text{SEM}\,|\,\text{CONT}\,|\,\text{RES } \boxed{1} \\ \text{C-DTRS } \boxed{2} \end{bmatrix} \end{bmatrix}.$$

Two Problems

As an 'almost' description, part 2 of the Semantics Principle reads

$$((:\sim\textit{phrasal-sign}\wedge:\text{DTRS}\sim\textit{headed-structure})$$
$$\rightarrow:\text{SEM INDS}\approx\text{collect-indices}(:\text{DTRS})).$$

Again, the function collect-indices is not part of our descriptive formalism. Again too, we can incorporate collect-indices into our formalism by creating new objects out of old objecst and constraining the components of these new objects with a theory. However, we must address two issues before we begin this incorporation. The first concerns sets membership, and the second concerns constituent structures.

A descriptive formalism cannot express set membership. Certainly, a set is an object, but we cannot arbitrarily say that an object is a member of a set. Consequently, we cannot arbitrarily say that a set is a subset of another set, or that a set is the union of other sets. Unfortunately, the collect-indices function requires that we state a given set is the union of other sets. Fortunately, HPSG only seems to need finite sets, and we can just as easily use lists in place of finite sets. An object is a 'member' of a list iff it is the FIRST of that list or it is a 'member' of the REST of that list. A first list is a 'subset' of a second list iff the FIRST of the first list is a 'member' of the second list and the REST of the first list is a 'subset' of the second list. More germane to us here, the 'union' of two lists is their concatenation. We will work with lists of indices rather than sets of indices to capture combine-semantics, and, to paraphrase [*Pollard and Sag 1987*],

> collect-indices is just the function defined on feature structures of type *constituent-structure* which takes the *concatenation* of the INDS values on all the daughters.

This still leaves the problem of constituent structures. Part 2 of the semantics principle says that if a phrase has a headed structure, then the phrase's semantic indices are the collect-indices of the phrase's daughters. Unfortunately, different kinds of headed structure store daughters in different places. For example, a head filler structure only stores two daughters, one in its H-DTR and one in its F-DTR. (A head filler structure also has a C-DTRS, but this is always an empty list, so it provides no daughters from which to collect indices.) On the other hand, a head complement structure stores one daughter in its H-DTR, but stores a list of daughters in its C-DTRS. Part 2 of the Semantics Principle must "know" where a given subtype of headed structure stores daughters in order to find them and collect their indices together. Consequently, we must write a Semantics Principle (Part 2) for each minimal subtype of *headed-structure*.

This is not as bad as it sounds. Headed structures only store daughters in two ways: as a single daughter (i.e. the F-DTR of a head filler structure), or as a list of daughters (i.e. the C-DTRS of a head complement structure). For this handout, assume *head-filler-structure* and *head-complement-structure* are the only subtypes of *headed-structure*. I will illustrate the collection of indices from stores that hold a single daughter and from stores that hold a list of daughters by writing Semantics Principles (Part 2) for phrases with head filler structures and for phrases with head complement structures.

Appends of Indices

Lists of indices are exactly analogous to lists of signs. Introduce type *aoi* and subtypes *faoi* and *uaoi*. *faoi* and *uaoi* have no subtypes, so they are both varieties. Let

$$\mathfrak{S}\langle \mathit{faoi}, \mathrm{RES}\rangle = \{\mathit{eloi}, \mathit{neloi}\},$$

$$\mathfrak{S}\langle \mathit{faoi}, \mathrm{LEFT}\rangle = \{\mathit{eloi}\},$$

$$\mathfrak{S}\langle \mathit{faoi}, \mathrm{RIGHT}\rangle = \{\mathit{eloi}, \mathit{neloi}\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle \mathit{faoi}, a\rangle = \emptyset.$$

Let

$$\mathfrak{S}\langle \mathit{uaoi}, \mathrm{RES}\rangle = \{\mathit{eloi}, \mathit{neloi}\},$$

$$\mathfrak{S}\langle \mathit{uaoi}, \mathrm{LEFT}\rangle = \{\mathit{neloi}\},$$

$$\mathfrak{S}\langle \mathit{uaoi}, \mathrm{RIGHT}\rangle = \{\mathit{eloi}, \mathit{neloi}\},$$

$$\mathfrak{S}\langle \mathit{uaoi}, \mathrm{SUB}\rangle = \{\mathit{faoi}, \mathit{uaoi}\},$$

and for all other $a \in \mathfrak{A}$,

$$\mathfrak{S}\langle \mathit{uaoi}, a\rangle = \emptyset.$$

Let

$$d_4 = (:\sim \mathit{faoi} \rightarrow :\mathrm{RES}\approx:\mathrm{RIGHT}), \text{ and}$$

$$\begin{aligned} d_5 = (:\sim \mathit{uaoi} \rightarrow (&:\mathrm{RES\ FIRST}\approx:\mathrm{LEFT\ FIRST} \\ &\wedge:\mathrm{SUB\ RES}\approx:\mathrm{RES\ REST} \\ &\wedge:\mathrm{SUB\ LEFT}\approx:\mathrm{LEFT\ REST} \\ &\wedge:\mathrm{SUB\ RIGHT}\approx:\mathrm{RIGHT})). \end{aligned}$$

As AVMs

$$d_4 = \mathit{faoi}[\,] \rightarrow \begin{bmatrix} \mathrm{RES}\ \boxed{1} \\ \mathrm{RIGHT}\ \boxed{1} \end{bmatrix},$$

$$d_5 = \mathit{uaoi}[\,] \rightarrow \begin{bmatrix} \mathrm{RES} & \begin{bmatrix} \mathrm{FIRST}\ \boxed{1} \\ \mathrm{REST}\ \boxed{2} \end{bmatrix} \\ \mathrm{LEFT} & \begin{bmatrix} \mathrm{FIRST}\ \boxed{1} \\ \mathrm{REST}\ \boxed{3} \end{bmatrix} \\ \mathrm{RIGHT}\ \boxed{4} & \\ \mathrm{SUB} & \begin{bmatrix} \mathrm{RES}\ \boxed{2} \\ \mathrm{LEFT}\ \boxed{3} \\ \mathrm{RIGHT}\ \boxed{4} \end{bmatrix} \end{bmatrix}.$$

Suppose that structure S is a model of $\{d_4, d_5\}$ and that u is an append of indices in S. Then u's RES is the concatenation of u's LEFT and u's RIGHT.

The Semantics Principle (Part 2) for Head Filler Structures

Let the SEM | INDS of a sign be an append of indices, and let the list of indices previously considered to be a sign's SEM | INDS now be its SEM | INDS | RES. Thus, the Semantics Principle (Part 2) for Head Filler Structures reads, as a description,

$$\begin{aligned} d_{hfs} = (&(:\sim \textit{phrasal-sign} \wedge :\text{DTRS} \sim \textit{head-filler-structure}) \\ &\rightarrow (:\text{SEM INDS LEFT} \approx :\text{DTRS H-DTR SEM INDS RES} \\ &\quad \wedge :\text{SEM INDS RIGHT} \approx :\text{DTRS F-DTR SEM INDS RES})), \end{aligned}$$

or, as an AVM,

$$d_{hfs} = \textit{phrasal-sign}\left[\text{DTRS } \textit{head-filler-structure}[\;]\right]$$
$$\rightarrow \begin{bmatrix} \text{SEM} \mid \text{IND} \begin{bmatrix} \text{LEFT } \boxed{1} \\ \text{RIGHT } \boxed{2} \end{bmatrix} \\ \text{DTRS} \begin{bmatrix} \text{H-DTR} \mid \text{SEM} \mid \text{INDS} \mid \text{RES } \boxed{1} \\ \text{F-DTR} \mid \text{SEM} \mid \text{INDS} \mid \text{RES } \boxed{2} \end{bmatrix} \end{bmatrix}.$$

Complement Daughters

Create a new kind of object called a complement daughters, and introduce the type *cd* for such objects. Each complement daughter is either finished or unfinished. Introduce subtypes *fcd* and *ucd* of *cd*. Finished complement daughters are of type *fcd*, and unfinished complement daughters are of type *ucd*. *fcd* and *ucd* have no subtypes, so they are both varieties. Let

$\mathfrak{S}\langle \textit{fcd}, \text{AOI}\rangle = \{\textit{faoi}\}$,

$\mathfrak{S}\langle \textit{fcd}, \text{LOS}\rangle = \{\textit{elos}\}$,

and for all other $a \in \mathfrak{A}$,

$\mathfrak{S}\langle \textit{fcd}, a\rangle = \emptyset$.

Let

$\mathfrak{S}\langle \textit{ucd}, \text{AOI}\rangle = \{\textit{faoi}, \textit{uaoi}\}$,

$\mathfrak{S}\langle \textit{ucd}, \text{LOS}\rangle = \{\textit{nelos}\}$,

$\mathfrak{S}\langle \textit{ucd}, \text{SUB}\rangle = \{\textit{fcd}, \textit{ucd}\}$,

and for all other $a \in \mathfrak{A}$,

$\mathfrak{S}\langle \textit{ucd}, a\rangle = \emptyset$.

Let

$d_6 = (:\!\sim\! \mathit{fcd} \!\rightarrow\! :\mathrm{AOI\ RES} \!\sim\! \mathit{eloi})$, and

$d_7 = (:\!\sim\! \mathit{ucd} \!\rightarrow\! (:\mathrm{AOI\ LEFT} \approx :\mathrm{LOS\ FIRST\ SEM\ INDS\ RES}$
$\quad \wedge :\mathrm{AOI\ RIGHT} \approx :\mathrm{SUB\ AOI\ RES}$
$\quad \wedge :\mathrm{LOS\ REST} \approx :\mathrm{SUB\ LOS}))$.

As AVMs,

$d_6 = \mathit{fcd}[\,] \rightarrow [\,\mathrm{AOI} \mid \mathrm{RES}\ \mathit{eloi}[\,]\,]$, and

$$d_7 = \mathit{ucd}[\,] \rightarrow \begin{bmatrix} \mathrm{AOI} & \begin{bmatrix} \mathrm{LEFT}\ \boxed{1} \\ \mathrm{RIGHT}\ \boxed{2} \end{bmatrix} \\ \mathrm{LOS} & \begin{bmatrix} \mathrm{FIRST} \mid \mathrm{SEM} \mid \mathrm{INDS} \mid \mathrm{RES}\ \boxed{1} \\ \mathrm{REST}\ \boxed{3} \end{bmatrix} \\ \mathrm{SUB} & \begin{bmatrix} \mathrm{LOS}\ \boxed{3} \\ \mathrm{AOI} \mid \mathrm{RES}\ \boxed{2} \end{bmatrix} \end{bmatrix}.$$

Suppose that structure S is a model of $\{d_4, d_5, d_6, d_7\}$ and that u is a complement daughters in S. Then u's AOI | RES is the 'union' of the SEM | INDS | RES's of the signs in u's LOS.

The Semantics Principle (Part 2) for Head Complement Structures

Let the C-DTRS of a head complement structure be a complement daughters, and let the list of signs previously considered to be the C-DTRS of a head complement structure now be the C-DTRS | LOS of a head complement structure. Thus, the Semantics Principle (Part 2) for Head Complement Structures reads, as a description,

$d_{hcs} = ((:\!\sim\! \mathit{phrasal\text{-}sign} \wedge :\mathrm{DTRS} \!\sim\! \mathit{head\text{-}complement\text{-}structure})$
$\quad \rightarrow (:\mathrm{SEM\ INDS\ LEFT} \approx :\mathrm{DTRS\ H\text{-}DTR\ SEM\ INDS\ RES}$
$\quad\quad \wedge :\mathrm{SEM\ INDS\ RIGHT} \approx :\mathrm{DTRS\ C\text{-}DTRS\ AOI\ RES}))$,

or, as an AVM,

$$d_{hfs} = \mathit{phrasal\text{-}sign}\,[\,\mathrm{DTRS}\ \mathit{head\text{-}complement\text{-}structure}[\,]\,]$$
$$\rightarrow \begin{bmatrix} \mathrm{SEM} \mid \mathrm{IND} & \begin{bmatrix} \mathrm{LEFT}\ \boxed{1} \\ \mathrm{RIGHT}\ \boxed{2} \end{bmatrix} \\ \mathrm{DTRS} & \begin{bmatrix} \mathrm{H\text{-}DTR} \mid \mathrm{SEM} \mid \mathrm{INDS} \mid \mathrm{RES}\ \boxed{1} \\ \mathrm{C\text{-}DTRS} \mid \mathrm{AOI} \mid \mathrm{RES}\ \boxed{2} \end{bmatrix} \end{bmatrix}.$$

Language Specific Principles

Constituent Ordering Principles

A constituent ordering principle primarily relates the phonology of a phrasal sign to the phonologies of the constituents of that phrasal sign. For example,

> "in any phrasal sign of English, (the phonological realization of) a daughter which is a lexical head is constrained to temporally precede (the phonological realization of) *any* of its sisters. In short, lexical heads in English are phrase-initial." [*Pollard and Sag 1987*, page 172]

Introduce appends of phons in exactly the same way that we introduced appends of signs and appends of indices. Introduce type *aop* with varietal subtypes *faop* and *uaop*. Write descriptions d_{faop} and d_{uaop} such that for each append of phons u in a model S of $\{d_{faop}, d_{uaop}\}$, the RES of u is the concatenation of the LEFT of u and the RIGHT of u. Let the PHON of a sign be an append of phons. Then the above constituent ordering principle can be expressed as the description

((:∼*phrasal-sign*

∧:DTRS∼*headed-structure*

∧:DTRS H-DTR∼*lexical-sign*)

→:PHON LEFT≈:DTRS H-DTR PHON RES),

or the AVM

$$\textit{phrasal-sign}\left[\text{DTRS }\textit{headed-structure}\left[\text{H-DTR }\textit{lexical-sign}[\;]\right]\right]$$

$$\rightarrow \begin{bmatrix}\text{PHON}\,|\,\text{LEFT}\;\boxed{1}\\ \text{DTRS}\,|\,\text{H-DTR}\,|\,\text{PHON}\,|\,\text{RES}\;\boxed{1}\end{bmatrix}.$$

Grammar and Lexicon Principles

A language has a finite number of constituent ordering principles, but it has exactly one grammar principle and exactly one lexicon principle. The grammar principle gives the legitimate options for a phrasal sign of the language, and the lexicon principle gives the legitimate options for a lexical sign of the language. A grammar principle has the form

(:∼*phrasal-sign*→$(G_1 \vee \ldots \vee G_m)$)

and a lexicon principle has the form

(:∼*lexical-sign*→$(L_1 \vee \ldots \vee L_n)$),

where $G_1, \ldots, G_m$ are the options for a phrasal sign, and $L_1, \ldots, L_n$ are the options for a lexical sign. Typically, $G_1, \ldots, G_m$ are called grammar rules, and $L_1, \ldots, L_n$ are called lexical entries.

For example, HPSG1 claims that

> "one of the possibilities for a phrasal sign in English is to be a saturated (i.e. [SUBCAT⟨ ⟩]) sign which has as constituents a single complement daughter (i.e. [C-DTRS⟨[ ]⟩]) and a head daughter, which in turn is constrained to be a phrasal rather than a lexical sign." [*Pollard and Sag 1987*, page 149]

Pollard and Sag also express this grammar rule as the AVM

$$\begin{bmatrix} \mathrm{SYN} \mid \mathrm{LOC} \mid \mathrm{SUBCAT}\ \langle\ \rangle \\ \mathrm{DTRS} \begin{bmatrix} \mathrm{H\text{-}DTR} \mid \mathrm{SYN} \mid \mathrm{LOC} \mid \mathrm{LEX}\ - \\ \mathrm{C\text{-}DTRS}\ \langle [\ ] \rangle \end{bmatrix} \end{bmatrix}.$$

We can express this grammar rule as the description

(:SYN LOC SUBCAT RES∼*elos*

∧:DTRS∼*head-complement-structure*

∧:DTRS H-DTR SYN LOC LEX∼−

∧:DTRS C-DTRS LOS REST∼*elos*).

Entailment

Fix a signature $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ throughout this handout.

Entailment

$T \subseteq [\![\mathfrak{S}]\!]$ entails $d \in [\![\mathfrak{S}]\!]$ iff for each $\mathfrak{S}$-structure S, $[\![T]\!]_S \subseteq [\![d]\!]_S$. Write $T \models d$ for T entails d. Write $d_1 \models d_2$ for $\{d_1\} \models d_2$. Write $\models d$ for $\emptyset \models d$.

Proposition 1

If $T \subseteq [\![\mathfrak{S}]\!]$ *and* d_1 *and* $d_2 \in [\![\mathfrak{S}]\!]$ *then*

$T \cup \{d_1\} \models d_2$ *iff* $T \models (d_1 \to d_2)$.

Proof

$T \cup \{d_1\} \models d_2$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$,
if $u \in [\![T \cup \{d_1\}]\!]_S$ then $u \in [\![d_2]\!]_S$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$,
if $u \in [\![T]\!]_S$ and $u \in [\![d_1]\!]_S$ then $u \in [\![d_2]\!]_S$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$,
if $u \in [\![T]\!]_S$ then (if $u \in [\![d_1]\!]_S$ then $u \in [\![d_2]\!]_S$)

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$,
if $u \in [\![T]\!]_S$ then $u \in [\![(d_1 \to d_2)]\!]_S$

$\Leftrightarrow T \models (d_1 \to d_2)$.

$\square$

Satisfiability

$T \subseteq [\![\mathfrak{S}]\!]$ is unsatisfiable iff for each $\mathfrak{S}$-structure S, $[\![T]\!]_S = \emptyset$.

Proposition 2

If $T \subseteq [\![\mathfrak{S}]\!]$ *and* $d \in [\![\mathfrak{S}]\!]$ *then*

$T \models d$ *iff* $T \cup \{\neg d\}$ *is unsatisfiable, and*

$T \models \neg d$ *iff* $T \cup \{d\}$ *is unsatisfiable.*

Proof

I only prove that

$T \models d$ iff $T \cup \{\neg d\}$ is unsatisfiable.

Proving that

$T \models \neg d$ iff $T \cup \{d\}$ is unsatisfiable

is exactly analogous.

$T \models d$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$, if $u \in [\![T]\!]_S$ then $u \in [\![d]\!]_S$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$, if $u \in [\![T]\!]_S$ then $u \notin [\![\neg d]\!]_S$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$, $u \notin [\![T]\!]_S$ or $u \notin [\![\neg d]\!]_S$

$\Leftrightarrow$ for each $\mathfrak{S}$-structure $S = \langle U, V, A \rangle$ and each $u \in U$, $u \notin [\![T \cup \{\neg d\}]\!]_S$

$\Leftrightarrow$ $[\![T \cup \{\neg d\}]\!]_S$ is unsatisfiable.

$\square$

Derivation

Fix a signature $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ throughout this handout.

Axioms

An **$\mathfrak{S}$-axiom** is a member of the smallest set X such that

A1. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $(d_1{\to}(d_2{\to}d_1)) \in X$,

A2. if $d_1, d_2, d_3 \in [\![\mathfrak{S}]\!]$
then $((d_1{\to}(d_2{\to}d_3)){\to}((d_1{\to}d_2){\to}(d_1{\to}d_3))) \in X$,

A3. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $((\neg d_1{\to}\neg d_2){\to}(d_2{\to}d_1)) \in X$,

A4. if $d \in [\![\mathfrak{S}]\!]$ then $(\neg\neg d{\to}d) \in X$,

A5. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $(d_1{\to}(d_2{\to}(d_1{\wedge}d_2))) \in X$,

A6. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $((d_1{\wedge}d_2){\to}d_1) \in X$,

A7. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $((d_1{\wedge}d_2){\to}d_2) \in X$,

A8. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $(d_1{\to}(d_1{\vee}d_2)) \in X$,

A9. if $d_1, d_2 \in [\![\mathfrak{S}]\!]$ then $(d_2{\to}(d_1{\vee}d_2)) \in X$,

A10. if $d_1, d_2, d_3 \in [\![\mathfrak{S}]\!]$
then $((d_1{\to}d_3){\to}((d_2{\to}d_3){\to}((d_1{\vee}d_2){\to}d_3))) \in X$,

A11. if $t_1, t_2 \in [\mathfrak{S}]$ then $(t_1{\approx}t_2{\to}t_2{\approx}t_1) \in X$,

A12. if $t_1, t_2, t_3 \in [\mathfrak{S}]$ then $(t_1{\approx}t_2{\to}(t_2{\approx}t_3{\to}t_1{\approx}t_3)) \in X$,

A13. if $t_1, t_2 \in [\mathfrak{S}]$ and $a \in \mathfrak{A}$
then $(t_1a{\approx}t_1a{\to}(t_1{\approx}t_2{\to}t_1a{\approx}t_2a)) \in X$,

A14. if $t \in [\mathfrak{S}]$ and $a \in \mathfrak{A}$ then $(ta{\approx}ta{\to}t{\approx}t) \in X$,

A15. $\top \in X$,

A16. if $t \in [\mathfrak{S}]$ and $v \in \mathfrak{V}$ then $(t{\sim}v{\to}t{\approx}t) \in X$,

A17. if $t \in [\mathfrak{S}]$ then $(t{\approx}t{\to}\bigvee\{t{\sim}v \mid v \in \mathfrak{V}\}) \in X$,

A18. if $t \in [\mathfrak{S}]$, $v_1, v_2 \in \mathfrak{V}$ and $v_1 \neq v_2$
then $(t{\sim}v_1{\to}(t{\sim}v_2{\to}\bot)) \in X$,

A19. if $t_1, t_2 \in [\mathfrak{S}]$ and $v \in \mathfrak{V}$
then $(t_1{\approx}t_2{\to}(t_1{\sim}v{\to}t_2{\sim}v)) \in X$,

A20. if $t \in [\mathfrak{S}]$, $v_1, v_2 \in \mathfrak{V}$, $a \in \mathfrak{A}$ and $v_2 \notin \mathfrak{S}\langle v_1, a\rangle$
then $(t{\sim}v_1{\to}(ta{\sim}v_2{\to}\bot)) \in X$, and

A21. if $t \in [\mathfrak{S}]$, $v \in \mathfrak{V}$, $a \in \mathfrak{A}$ and $\mathfrak{S}\langle v, a\rangle \neq \emptyset$
then $(t{\sim}v{\to}ta{\approx}ta) \in X$.

Write $\langle\!\langle\mathfrak{S}\rangle\!\rangle$ for the set of $\mathfrak{S}$-axioms.

Derivation

An $\mathfrak{S}$**-derivation** from $T \subseteq [\![\mathfrak{S}]\!]$ is a finite sequence $\langle d_1, \ldots, d_n \rangle \in [\![\mathfrak{S}]\!]^\star$ where for each $1 \leq k \leq n$,

$d_k \in \langle\!\langle \mathfrak{S} \rangle\!\rangle$,

$d_k \in T$, or

for some $1 \leq i < k$ and $1 \leq j < k$, $d_i = (d_j \rightarrow d_k)$.[2]

I often write derivation $\langle d_1, \ldots, d_n \rangle$ as

1	d_1	c_1
$\vdots$	$\vdots$	$\vdots$
n	d_n	c_n

where for each $1 \leq k \leq n$,

c_k is A1 if d_k is an axiom of the form A1,

$\vdots$

c_k is A21 if d_k is an axiom of the form A21,

c_k is T if $d_k \in T$, and

c_k is (i,j) if for some $1 \leq i < k$ and $1 \leq j < k$, $d_i = (d_j \rightarrow d_k)$.

$T \subseteq [\![\mathfrak{S}]\!]$ **derives** $d \in [\![\mathfrak{S}]\!]$ iff for some $\mathfrak{S}$-derivation $\langle d_1, \ldots, d_n \rangle$ from T, $d_n = d$. Write $T \vdash d$ for T derives d. Write $d_1 \vdash d_2$ for $\{d_1\} \vdash d_2$. Write $\vdash d$ for $\emptyset \vdash d$.

Proposition 1

If $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure and $d \in \langle\!\langle \mathfrak{S} \rangle\!\rangle$ then $[\![d]\!]_S = U$.

Proof

We prove 21 cases, one for each axiom 'shape'!

A1. For each $u \in U$,

$u \in [\![d_1]\!]_S$

$\Rightarrow$ if $u \in [\![d_2]\!]_S$ then $u[\![d_1]\!]_S$

$\Rightarrow u \in [\![(d_2 \rightarrow d_1)]\!]_S$.

Thus, for each $u \in U$,

$u \in [\![(d_1 \rightarrow (d_2 \rightarrow d_1))]\!]_S$.

A2—10 have proofs similar to that for A1.

A11. For each $u \in U$,

$u \in [\![t_1 \approx t_2]\!]_S$

[2] In this last case, we say that d_k comes from d_i and d_j by *modus ponens*.

$\Rightarrow [t_1]_S$ is defined on u, $[t_2]_S$ is defined on u, and $[t_1]_S(u) = [t_2]_S(u)$

$\Rightarrow u \in [\![t_2 {\approx} t_1]\!]_S$.

Thus, for each $u \in U$,

$u \in [\![(t_1 {\approx} t_2 {\to} t_2 {\approx} t_1)]\!]_S$.

A12—15 have proofs similar to that for A11.

A16. For each $u \in U$,

$u \in [\![t {\sim} v]\!]_S$

$\Rightarrow [t]_S$ is defined on u

$\Rightarrow u \in [\![t {\approx} t]\!]_S$.

Thus, for each $u \in U$,

$u \in [\![(t {\sim} v {\to} t {\approx} t)]\!]_S$.

A17. For each $u \in U$,

$u \in [\![t {\approx} t]\!]_S$

$\Rightarrow [t]_S$ is defined on u

$\Rightarrow [t]_S$ is defined on u, and for some $v \in \mathfrak{V}$, $V([t]_S(u)) = v$

$\Rightarrow$ for some $v \in \mathfrak{V}$, $u \in [\![t {\sim} v]\!]_S$

$\Rightarrow u \in [\![\bigvee\{t {\sim} v \mid v \in \mathfrak{V}\}]\!]_S$.

Thus, for each $u \in U$,

$u \in [\![(t {\approx} t {\to} \bigvee\{t {\sim} v \mid v \in \mathfrak{V}\})]\!]_S$.

A18. If $v_1 \neq v_2$ then for each $u \in U$,

$u \in [\![t {\sim} v_1]\!]_S$

$\Rightarrow [t]_S$ is defined on u, and $V([t]_S(u)) = v_1$

$\Rightarrow (u \in [\![t {\sim} v_2]\!]_S$

$\Rightarrow [t]_S$ is defined on u, $V([t]_S(u)) = v_1$, and $V([t]_S(u)) = v_2$

$\Rightarrow v_1 = v_2$

$\Rightarrow$ contradiction

$\Rightarrow u \in [\![\bot]\!]_S)$

$\Rightarrow u \in [\![(t {\sim} v_2 {\to} \bot)]\!]_S$.

Thus, if $v_1 \neq v_2$ then for each $u \in U$,

$u \in [\![(t {\sim} v_1 {\to} (t {\sim} v_2 {\to} \bot))]\!]_S$.

A19. For each $u \in U$,

$u \in [\![t_1 {\approx} t_2]\!]_S$

$\Rightarrow [t_1]_S$ is defined on u, $[t_2]_S$ is defined on u, and $[t_1]_S(u) = [t_2]_S(u)$

$\Rightarrow (u \in [\![t_1{\sim}v]\!]_S$

$\Rightarrow [t_1]_S$ is defined on u, $[t_2]_S$ is defined on u, $[t_1]_S(u) = [t_2]_S(u)$, and $V([t_1]_S(u)) = v$

$\Rightarrow [t_2]_S$ is defined on u, and $V([t_1]_S(u)) = v$

$\Rightarrow u \in [\![t_2{\sim}v]\!]_S)$

$\Rightarrow u \in [\![(t_1{\sim}v{\rightarrow}t_2{\sim}v)]\!]_S$.

Thus, for each $u \in U$,

$u \in [\![(t_1{\approx}t_2{\rightarrow}(t_1{\sim}v{\rightarrow}t_2{\sim}v))]\!]_S$.

A20. If $v_2 \notin \mathfrak{S}\langle v_1, a\rangle$ then for each $u \in U$,

$u \in [\![t{\sim}v_1]\!]_S$

$\Rightarrow [t]_S$ is defined on u, and $V([t]_S(u)) = v_1$

$\Rightarrow (u \in [\![ta{\sim}v_2]\!]_S$

$\Rightarrow [t]_S$ is defined on u, $V([t]_S(u)) = v_1$, $[ta]_S$ is defined on u, and $V([ta]_S(u)) = v_2$

$\Rightarrow [t]_S$ is defined on u, $V([t]_S(u)) = v_1$, $A(a)$ is defined on $[t]_S(u)$, and $V(A(a)([t]_S(u))) = v_2$

$\Rightarrow v_2 \in \mathfrak{S}\langle v_1, a\rangle$ by condition S1 of a structure

$\Rightarrow$ contradiction

$\Rightarrow u \in [\![\perp]\!]_S)$

$\Rightarrow u \in [\![(ta{\sim}v_2{\rightarrow}\perp)]\!]_S$.

Thus, if $v_2 \notin \mathfrak{S}\langle v_1, a\rangle$ then for each $u \in U$,

$u \in [\![(t{\sim}v_1{\rightarrow}(ta{\sim}v_2{\rightarrow}\perp))]\!]_S$.

A21. If $\mathfrak{S}\langle v, a\rangle \neq \emptyset$ then for each $u \in U$,

$u \in [\![t{\sim}v]\!]_S$

$\Rightarrow [t]_S$ is defined on u, and $V([t]_S(u)) = v$

$\Rightarrow [t]_S$ is defined on u, and $A(a)$ is defined on $[t]_S(u)$ by condition S2 of a structure

$\Rightarrow [ta]_S$ is defined on u

$\Rightarrow u \in [\![ta{\approx}ta]\!]_S$.

Thus, if $\mathfrak{S}\langle v, a\rangle \neq \emptyset$ then for each $u \in U$,

$u \in [\![(t{\sim}v{\rightarrow}ta{\approx}ta)]\!]_S$.

□

Lemma 2

If $T \cup \{d, d', d''\} \subseteq [\![\mathfrak{S}]\!]$, $T \vdash (d{\rightarrow}(d'{\rightarrow}d''))$, and $T \vdash (d{\rightarrow}d')$ then $T \vdash (d{\rightarrow}d'')$.

Proof

$T \vdash (d{\rightarrow}(d'{\rightarrow}d''))$ and $T \vdash (d{\rightarrow}d')$

$\Rightarrow$ for some $d_1, \ldots, d_{m-1}, d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d_1, \ldots, d_{m-1}, (d{\rightarrow}(d'{\rightarrow}d''))\rangle$ and $\langle d'_1, \ldots, d'_{n-1}, (d{\rightarrow}d')\rangle$
are $\mathfrak{S}$-derivations from T

$\Rightarrow$ for some $d_1, \ldots, d_{m-1}, d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,

1	d_1	
$\vdots$	$\vdots$	
$m-1$	d_{m-1}	
m	$(d{\rightarrow}(d'{\rightarrow}d''))$	
$m+1$	d'_1	
$\vdots$	$\vdots$	
$m+n-1$	d'_{n-1}	
$m+n$	$(d{\rightarrow}d')$	
$m+n+1$	$((d{\rightarrow}(d'{\rightarrow}d'')){\rightarrow}((d{\rightarrow}d'){\rightarrow}(d{\rightarrow}d'')))$	A2
$m+n+2$	$((d{\rightarrow}d'){\rightarrow}(d{\rightarrow}d''))$	$(m+n+1, m)$
$m+n+3$	$(d{\rightarrow}d'')$	$(m+n+2, m+n)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash (d{\rightarrow}d'')$.

□

Lemma 3

If $\langle d_1, \ldots, d_k\rangle$ is an $\mathfrak{S}$-derivation from $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$ then $T \vdash (d{\rightarrow}d_k)$.

Proof

By induction on k.

Case 1.

$d_k \in \langle\!\{\mathfrak{S}\}\!\rangle \cup T$

$\Rightarrow$	1	$(d_k{\rightarrow}(d{\rightarrow}d_k))$	A1
	2	d_k	A1, …, A21 or T
	3	$(d{\rightarrow}d_k)$	$(1, 2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash (d \rightarrow d_k)$.

Case 2.

$d_k = d$

$\Rightarrow$	1	$((d \rightarrow ((d \rightarrow d) \rightarrow d)) \rightarrow ((d \rightarrow (d \rightarrow d)) \rightarrow (d \rightarrow d_k)))$	A2
	2	$(d \rightarrow ((d \rightarrow d) \rightarrow d))$	A1
	3	$((d \rightarrow (d \rightarrow d)) \rightarrow (d \rightarrow d_k))$	$(1, 2)$
	4	$(d \rightarrow (d \rightarrow d))$	A1
	5	$(d \rightarrow d_k)$	$(3, 4)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash (d \rightarrow d_k)$.

Case 3.

for some $1 \leq i < k$ and $1 \leq j < k$, $d_i = (d_j \rightarrow d_k)$

$\Rightarrow$ for some $1 \leq i < k$ and $1 \leq j < k$,
$d_i = (d_j \rightarrow d_k)$, $T \vdash (d \rightarrow d_i)$, and $T \vdash (d \rightarrow d_j)$ by induction

$\Rightarrow$ for some $1 \leq j < k$, $T \vdash (d \rightarrow (d_j \rightarrow d_k))$ and $T \vdash (d \rightarrow d_j)$

$\Rightarrow T \vdash (d \rightarrow d_k)$, by Lemma 2

□

Proposition 4 (The Deduction Lemma)

If $T \cup \{d_1, d_2\} \subseteq [\![\mathfrak{S}]\!]$ *then*

$T \cup \{d_1\} \vdash d_2$ *iff* $T \vdash (d_1 \rightarrow d_2)$.

Proof

From left to right.

$T \cup \{d_1\} \vdash d_2$

$\Rightarrow$ for some $d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d'_1, \ldots, d'_{n-1}, d_2 \rangle$ is an $\mathfrak{S}$-derivation from $T \cup \{d_1\}$

$\Rightarrow T \vdash (d_1 \rightarrow d_2)$. by Lemma 3

From right to left.

$T \vdash (d_1 \rightarrow d_2)$

$\Rightarrow$ for some $d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d'_1, \ldots, d'_{n-1}, (d_1 \rightarrow d_2) \rangle$ is an $\mathfrak{S}$-derivation from T

$\Rightarrow$ for some $d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d'_1, \ldots, d'_{n-1}, (d_1 \rightarrow d_2) \rangle$ is an $\mathfrak{S}$-derivation from $T \cup \{d_1\}$

$\Rightarrow$ for some $d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,

1	d'_1	
$\vdots$	$\vdots$	
$n-1$	d'_{n-1}	
n	$(d_1 \rightarrow d_2)$	
$n+1$	d_1	$T \cup \{d_1\}$
$n+2$	d_2	$(n, n+1)$

is an $\mathfrak{S}$-derivation from $T \cup \{d_1\}$

$\Rightarrow T \cup \{d_1\} \vdash d_2$.

□

Consistency

$T \subseteq [\![\mathfrak{S}]\!]$ is inconsistent iff $T \vdash \bot$.

Proposition 5

If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$ then

$T \vdash d$ iff $T \cup \{\neg d\}$ is inconsistent, and

$T \vdash \neg d$ iff $T \cup \{d\}$ is inconsistent.

Proof

I only prove that

$T \vdash d$ iff $T \cup \{\neg d\}$ is inconsistent.

Proving that

$T \vdash \neg d$ iff $T \cup \{d\}$ is inconsistent

is exactly analogous.

From left to right.

$T \vdash d$

$\Rightarrow$ for some $d_1, \ldots, d_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d_1, \ldots, d_{n-1}, d\rangle$ is an $\mathfrak{S}$-derivation from T

$\Rightarrow$ for some $d_1, \ldots, d_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d_1, \ldots, d_{n-1}, d\rangle$ is an $\mathfrak{S}$-derivation from $T \cup \{\neg d\}$

$\Rightarrow$ for some $d_1, \ldots, d_{n-1} \in [\![\mathfrak{S}]\!]$,

1	d_1	
$\vdots$	$\vdots$	
$n-1$	d_{n-1}	
n	d	
$n+1$	$\neg d$	$T \cup \{\neg d\}$
$n+2$	$(\neg d \to (\neg\bot \to \neg d))$	$A1$
$n+3$	$(\neg\bot \to \neg d)$	$(n+2, n+1)$
$n+4$	$((\neg\bot \to \neg d) \to (d \to \bot))$	A3
$n+5$	$(d \to \bot)$	$(n+4, n+3)$
$n+5$	$\bot$	$(n+5, n)$

is an $\mathfrak{S}$-derivation from $T \cup \{\neg d\}$

$\Rightarrow T \cup \{\neg d\} \vdash \bot$

$\Rightarrow T \cup \{\neg d\}$ is inconsistent.

From right to left.

$T \cup \{\neg d\}$ is inconsistent.

$\Rightarrow T \cup \{\neg d\} \vdash \bot$

$\Rightarrow T \vdash (\neg d \to \bot)$ by Proposition 4

$\Rightarrow$ for some $d_1, \ldots, d_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d_1, \ldots, d_{n-1}, (\neg d \to \bot)\rangle$ is an $\mathfrak{S}$-derivation from T

$\Rightarrow$ for some $d_1, \ldots, d_{n-1} \in [\![\mathfrak{S}]\!]$,

1	d_1	
$\vdots$	$\vdots$	
$n-1$	d_{n-1}	
n	$(\neg d \to \bot)$	
$n+1$	$((\neg d \to \bot) \to (\top \to d))$	A3
$n+2$	$(\top \to d)$	$(n+1, n)$
$n+3$	$\top$	A15
$n+4$	d	$(n+2, n+3)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash d$.

□

Soundness

Lemma 1

If $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure, d_1 and $d_2 \in [\![\mathfrak{S}]\!]$, $u \in [\![(d_1 \to d_2)]\!]_S$, and $u \in [\![d_1]\!]_S$ then $u \in [\![d_2]\!]_S$.

Proof

$u \in [\![(d_1 \to d_2)]\!]_S$ and $u \in [\![d_1]\!]_S$

$\Rightarrow u \in (U \setminus [\![d_1]\!]_S) \cup [\![d_2]\!]_S$ and $u \in [\![d_1]\!]_S$

$\Rightarrow u \in [\![d_2]\!]_S$.

□

Lemma 2

If S is an $\mathfrak{S}$-structure, $T \cup \{d_1, \ldots, d_k\} \subseteq [\![\mathfrak{S}]\!]$, $\langle d_1, \ldots, d_k \rangle$ is an $\mathfrak{S}$-derivation from T, and $u \in [\![T]\!]_S$ then $u \in [\![d_k]\!]_S$.

Proof

By induction on k.

$\langle d_1, \ldots, d_k \rangle$ is an $\mathfrak{S}$-derivation from T

$\Rightarrow d_k \in \langle\!|\mathfrak{S}|\!\rangle$, $d_k \in T$, or
for some $1 \leq i < k$ and some $1 \leq j < k$, $d_i = (d_j \to d_k)$

$\Rightarrow u \in [\![d_k]\!]_S$, by Proposition 1 of "Derivation"
$u \in [\![d_k]\!]_S$, since $u \in [\![T]\!]_S$
or for some $1 \leq i < k$ and some $1 \leq j < k$,
$d_i = (d_j \to d_k)$, $u \in [\![d_i]\!]_S$ and $u \in [\![d_j]\!]_S$ by the inductive hypothesis

$\Rightarrow u \in [\![d_k]\!]_S$ or for some $1 \leq j < k$, $u \in [\![(d_j \to d_k)]\!]_S$ and $u \in [\![d_j]\!]_S$

$\Rightarrow u \in [\![d_k]\!]_S$. by Lemma 1

□

Lemma 3

If $S = \langle U, V, A \rangle$ is an $\mathfrak{S}$-structure, $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$, and $T \vdash d$ then $[\![T]\!]_S \subseteq [\![d]\!]_S$.

Proof

$T \vdash d$

$\Rightarrow$ for some $d_1, \ldots, d_{k-1} \in [\![\mathfrak{S}]\!]$,
$\langle d_1, \ldots, d_{k-1}, d \rangle$ is an $\mathfrak{S}$-derivation from T

$\Rightarrow$ for each $u \in U$, if $u \in [\![T]\!]_S$ then $u \in [\![d]\!]_S$ by Lemma 2

$\Rightarrow [\![T]\!]_S \subseteq [\![d]\!]_S$.

□

Theorem 4 (The Soundness Theorem)
If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$*, and* $T \vdash d$ *then* $T \models d$*.*

Proof

$T \vdash d$
$\Rightarrow$ for each $\mathfrak{S}$-structure S, $[\![T]\!]_S \subseteq [\![d]\!]_S$ by Lemma 3
$\Rightarrow T \models d$.

$\square$

Corollary 5
If $T \subseteq [\![\mathfrak{S}]\!]$*, and* T *is satisfiable then* T *is consistent.*

Proof

T is inconsistent
$\Rightarrow T \vdash \perp$
$\Rightarrow T \models \perp$ by Theorem 4
$\Rightarrow$ for each $\mathfrak{S}$-structure, $[\![T]\!]_S \subseteq [\![\perp]\!]_S$
$\Rightarrow$ for each $\mathfrak{S}$-structure, $[\![T]\!]_S = \emptyset$
$\Rightarrow T$ is unsatisfiable.

$\square$

Lindenbaum's Lemma

$T \subseteq [\![\mathfrak{S}]\!]$ is maximally consistent iff

T is consistent, and

for each $T' \subseteq [\![\mathfrak{S}]\!]$, if $T \subset T'$ then T' is inconsistent.

Lindenbaum's Lemma

If $T \subseteq [\![\mathfrak{S}]\!]$ is consistent then $T \subseteq T'$ for some maximally consistent $T' \subseteq [\![\mathfrak{S}]\!]$.

Proof

Let $[\![\mathfrak{S}]\!] = \{d_i \mid i < \delta\}$ for some ordinal δ. Construct a set of theories $\{T_i \mid i \leq \delta\}$ as follows. For each $i < \delta$, if

$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\}$ is consistent

then let

$T_i = (\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\}$,

and if

$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\}$ is inconsistent

then let

$T_i = (\cup\{T_j \mid j < i\}) \cup T$.

Let

$T_\delta = (\cup\{T_j \mid j < \delta\}) \cup T$.

I will show that T_δ is a maximally consistent superset of T. Clearly, $T \subseteq T_\delta$. It remains to show that T_δ is maximally consistent.

Claim

For each $i \leq \delta$, T_i is consistent.

Proof of Claim

By contradiction. Suppose T_i is inconsistent for some $i \leq \delta$. Let

$B = \{i \leq \delta \mid T_i$ *is inconsistent*$\}$.

Then B is nonempty. Let β be the smallest member of B. T_β is inconsistent. Therefore,

$T_\beta = (\cup\{T_j \mid j < \beta\}) \cup T$, *1*

and

for some $d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d'_1, \ldots, d'_{n-1}, \bot\rangle$ is an $\mathfrak{S}$-derivation from T_β.

Let $X = T_\beta \cap \{d'_1, \ldots, d'_{n-1}, \bot\}$. Then

X is finite, *2*

and

$\langle d'_1, \ldots, d'_{n-1}, \bot \rangle$ *is an* $\mathfrak{S}$*-derivation from* X. *3*

Let $Y = X \setminus T$. *For each* $d \in Y$, *let*

$\widehat{d} = \{i < \beta \mid d \in T_i\}$.

For each $d \in Y$, $\widehat{d}$ *is nonempty, since*

$d \in Y$

$\Rightarrow d \in X \setminus T$

$\Rightarrow d \in T_\beta \setminus T$

$\Rightarrow d \in \cup\{T_j \mid j < \beta\}$ *by 1*

$\Rightarrow$ *for some* $i < \beta$, $d \in T_i$

$\Rightarrow \widehat{d} \neq \emptyset$.

For each $d \in Y$, *let* $\widetilde{d}$ *be the smallest member of* $\widehat{d}$. *Let* $Z = \{\widetilde{d} \mid d \in Y\}$. Z *is finite, by 2. Also,* Z *is nonempty, since*

$Z = \emptyset$

$\Rightarrow Y = \emptyset$

$\Rightarrow X \subseteq T$

$\Rightarrow \langle d'_1, \ldots, d'_{n-1}, \bot \rangle$ *is an* $\mathfrak{S}$*-derivation from* T *by 3*

$\Rightarrow T \vdash \bot$

$\Rightarrow T$ *is inconsistent*

$\Rightarrow$ *contradiction.*

Let α *be the largest member of* Z. *Then*

$Y \subseteq T_\alpha$,

since

$d \in Y$

$\Rightarrow d \in T_{\widetilde{d}}$ and $\widetilde{d} \in Z$

$\Rightarrow d \in T_{\widetilde{d}}$ and $\widetilde{d} \leq \alpha$

$\Rightarrow d \in \cup\{T_i \mid i < \alpha\}$ *or* $d \in T_\alpha$

$\Rightarrow d \in T_\alpha$,

and

$T \subseteq T_\alpha$.

Therefore, $X \subseteq T_\alpha$, *since*

$d \in X$

$\Rightarrow d \in X \setminus T$ *or* $d \in T$

$\Rightarrow d \in Y$ *or* $d \in T$

$\Rightarrow d \in T_\alpha$.

Therefore, $\beta \le \alpha$*, since*

$X \subseteq T_\alpha$

$\Rightarrow \langle d'_1, \ldots, d'_{n-1}, \perp \rangle$ *is an* $\mathfrak{S}$*-derivation from* T_α *by 3*

$\Rightarrow T_\alpha$ *is inconsistent*

$\Rightarrow \alpha \in B$

$\Rightarrow \beta \le \alpha$.

However, $\alpha < \beta$*, since* $\alpha \in Z$ *and*

$i \in Z$

$\Rightarrow$ *for some* $d \in Y$*,* $i = \widetilde{d}$

$\Rightarrow$ *for some* $d \in Y$*,* $i \in \widehat{d}$

$\Rightarrow i < \beta$.

Contradiction. $\diamond$

By the claim, T_δ is consistent. Further, for each $T' \subseteq [\![\mathfrak{S}]\!]$,

$T_\delta \subset T'$

$\Rightarrow$ for some $i < \delta$, $d_i \in T' \setminus T_\delta$, and
for each $j < \delta$, $T_j \subseteq T'$

$\Rightarrow$ for some $i < \delta$, $d_i \in T' \setminus T_\delta$ and $T_i \subseteq T'$

$\Rightarrow$ for some $i < \delta$,
$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\} \not\subseteq T_\delta$ and
$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\} \subseteq T'$

$\Rightarrow$ for some $i < \delta$,
$T_i \neq (\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\}$ and
$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\} \subseteq T'$

$\Rightarrow$ for some $i < \delta$,
$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\}$ is inconsistent and
$(\cup\{T_j \mid j < i\}) \cup T \cup \{d_i\} \subseteq T'$

$\Rightarrow T'$ is inconsistent.

$\square$

Completeness

Lemma 1

If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$, and T is maximally consistent then

$d \in T$ *iff* $T \vdash d$.

Proof

From left to right. Clearly,

if $d \in T$ then $T \vdash d$.

From right to left.

$T \vdash d$ and $d \notin T$

$\Rightarrow T \vdash d$ and $T \cup \{d\}$ is inconsistent since T is maximally consistent

$\Rightarrow T \vdash d$ and $T \cup \{d\} \vdash \perp$

$\Rightarrow T \vdash d$ and $T \vdash (d \to \perp)$ by Proposition 4 of "Derivation"

$\Rightarrow$ for some $d_1, \ldots, d_{m-1}, d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,
$\langle d_1, \ldots, d_{m-1}, d \rangle$ and $\langle d'_1, \ldots, d'_{n-1}, (d \to \perp) \rangle$
are $\mathfrak{S}$-derivations from T

$\Rightarrow$ for some $d_1, \ldots, d_{m-1}, d'_1, \ldots, d'_{n-1} \in [\![\mathfrak{S}]\!]$,

$$\begin{array}{rll}
1 & d_1 & \\
\vdots & \vdots & \\
m-1 & d_{m-1} & \\
m & d & \\
m+1 & d'_1 & \\
\vdots & \vdots & \\
m+n-1 & d'_{n-1} & \\
m+n & (d \to \perp) & \\
m+n+1 & \perp & (m+n, m)
\end{array}$$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash \perp$

$\Rightarrow T$ is inconsistent

$\Rightarrow$ contradiction.

Thus,

if $T \vdash d$ then $d \in T$.

$\square$

Lemma 2

If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$, and T is maximally consistent then

$\neg d \in T$ *iff* $d \notin T$.

Proof

$\neg d \in T$

$\Leftrightarrow T \vdash \neg d$ — by Lemma 1

$\Leftrightarrow T \cup \{d\}$ is inconsistent — by Proposition 5 of "Derivation"

$\Leftrightarrow d \notin T$. — since T is maximally consistent

□

Lemma 3

If $T \cup \{d_1, d_2\} \subseteq [\![\mathfrak{S}]\!]$, and T is maximally consistent then

$(d_1 \wedge d_2) \in T$ *iff* $d_1 \in T$ *and* $d_2 \in T$.

Proof

From left to right.

$(d_1 \wedge d_2) \in T$

$\Rightarrow$

1	$((d_1 \wedge d_2) \to d_1)$	A6
2	$(d_1 \wedge d_2)$	T
3	d_1	$(1, 2)$

and

1	$((d_1 \wedge d_2) \to d_2)$	A7
2	$(d_1 \wedge d_2)$	T
3	d_2	$(1, 2)$

are $\mathfrak{S}$-derivations from T

$\Rightarrow T \vdash d_1$ and $T \vdash d_2$

$\Rightarrow d_1 \in T$ and $d_2 \in T$. — by Lemma 1

From right to left.

$d_1 \in T$ and $d_2 \in T$

$\Rightarrow$	1	$(d_1 \rightarrow (d_2 \rightarrow (d_1 \wedge d_2)))$	A5
	2	d_1	T
	3	$(d_2 \rightarrow (d_1 \wedge d_2))$	$(1, 2)$
	4	d_2	T
	5	$(d_1 \wedge d_2)$	$(3, 4)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash (d_1 \wedge d_2)$

$\Rightarrow (d_1 \wedge d_2) \in T$. by Lemma 1

□

Lemma 4

If $T \cup \{d_1, d_2\} \subseteq [\![\mathfrak{S}]\!]$*, and* T *is maximally consistent then*

$(d_1 \vee d_2) \in T$ *iff* $d_1 \in T$ *or* $d_2 \in T$*.*

Proof

From left to right.

$(d_1 \vee d_2) \in T$, $d_1 \notin T$ and $d_2 \notin T$

$\Rightarrow (d_1 \vee d_2) \in T$, $\neg d_1 \in T$ and $\neg d_2 \in T$ by Lemma 2

$\Rightarrow$	1	$((\neg\bot \rightarrow \neg d_1) \rightarrow (d_1 \rightarrow \bot))$	A3
	2	$(\neg d_1 \rightarrow (\neg\bot \rightarrow \neg d_1))$	A1
	3	$\neg d_1$	T
	4	$(\neg\bot \rightarrow \neg d_1)$	$(2, 3)$
	5	$(d_1 \rightarrow \bot)$	$(1, 4)$
	6	$((\neg\bot \rightarrow \neg d_2) \rightarrow (d_2 \rightarrow \bot))$	A3
	7	$(\neg d_2 \rightarrow (\neg\bot \rightarrow \neg d_2))$	A1
	8	$\neg d_2$	T
	9	$(\neg\bot \rightarrow \neg d_2)$	$(7, 8)$
	10	$(d_2 \rightarrow \bot)$	$(6, 9)$
	11	$((d_1 \rightarrow \bot) \rightarrow ((d_2 \rightarrow \bot) \rightarrow ((d_1 \vee d_2) \rightarrow \bot)))$	A10
	12	$((d_2 \rightarrow \bot) \rightarrow ((d_1 \vee d_2) \rightarrow \bot))$	$(11, 5)$
	13	$((d_1 \vee d_2) \rightarrow \bot)$	$(12, 10)$
	14	$(d_1 \vee d_2)$	T
	15	$\bot$	$(13, 14)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash \bot$

$\Rightarrow T$ is inconsistent

$\Rightarrow$ contradiction.

Thus,

if $(d_1 \vee d_2) \in T$ then $d_1 \in T$ or $d_2 \in T$.

From right to left.

$d_1 \in T$ or $d_2 \in T$

$\Rightarrow$	1	$(d_1 \rightarrow (d_1 \vee d_2))$	A8
	2	d_1	T
	3	$(d_1 \vee d_2)$	$(1,2)$
or			
	1	$(d_2 \rightarrow (d_1 \vee d_2))$	A9
	2	d_2	T
	3	$(d_1 \vee d_2)$	$(1,2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash (d_1 \vee d_2)$

$\Rightarrow (d_1 \vee d_2) \in T$. by Lemma 1

$\square$

Lemma 5

If $T \cup \{d_1, d_2\} \subseteq [\![\mathfrak{S}]\!]$, and T is maximally consistent then

$(d_1 \rightarrow d_2) \in T$ *iff* $d_1 \notin T$ *or* $d_2 \in T$.

Proof

From left to right.

$(d_1 \rightarrow d_2) \in T$ and $d_1 \in T$

$\Rightarrow$	1	$(d_1 \rightarrow d_2)$	T
	2	d_1	T
	3	d_2	$(1,2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash d_2$

$\Rightarrow d_2 \in T$. by Lemma 1

Thus,

if $(d_1 \rightarrow d_2) \in T$ then $d_1 \notin T$ or $d_2 \in T$.

From right to left.

$d_1 \notin T$ or $d_2 \in T$

$\Rightarrow \neg d_1 \in T$ or $d_2 \in T$ — by Lemma 2

$\Rightarrow$

1	$((\neg d_2 \to \neg d_1) \to (d_1 \to d_2))$	A3
2	$(\neg d_1 \to (\neg d_2 \to \neg d_1))$	A1
3	$\neg d_1$	T
4	$(\neg d_2 \to \neg d_1)$	$(2,3)$
5	$(d_1 \to d_2)$	$(1,4)$

or

1	$(d_2 \to (d_1 \to d_2))$	A1
2	d_2	T
3	$(d_1 \to d_2)$	$(1,2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash (d_1 \to d_2)$

$\Rightarrow (d_1 \to d_2) \in T.$ — by Lemma 1

□

Lemma 6

If $T \subseteq [\![\mathfrak{S}]\!]$, t_1 *and* $t_2 \in [\mathfrak{S}]$, *and* T *is maximally consistent then*

if $t_1 \approx t_2 \in T$ *then* $t_2 \approx t_1 \in T$.

Proof

$t_1 \approx t_2 \in T$

$\Rightarrow$

1	$(t_1 \approx t_2 \to t_2 \approx t_1)$	A11
2	$t_1 \approx t_2$	T
3	$t_2 \approx t_1$	$(1,2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash t_2 \approx t_1$

$\Rightarrow t_2 \approx t_1 \in T.$ — by Lemma 1

□

Lemma 7

If $T \subseteq [\![\mathfrak{S}]\!]$, t_1, t_2 *and* $t_3 \in [\mathfrak{S}]$, *and* T *is maximally consistent then*

if $t_1 \approx t_2 \in T$ *and* $t_2 \approx t_3 \in T$ *then* $t_1 \approx t_3 \in T$.

Proof

$t_1{\approx}t_2 \in T$ and $t_2{\approx}t_3 \in T$

$\Rightarrow$ 1	$(t_1{\approx}t_2 \rightarrow (t_2{\approx}t_3 \rightarrow t_1{\approx}t_3))$		A12
2	$t_1{\approx}t_2$		T
3	$(t_2{\approx}t_3 \rightarrow t_1{\approx}t_3)$		$(1,2)$
4	$t_2{\approx}t_3$		T
5	$t_1{\approx}t_3$		$(3,4)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash t_1{\approx}t_3$

$\Rightarrow t_1{\approx}t_3 \in T$. by Lemma 1

□

Lemma 8

If $T \subseteq [\![\mathfrak{S}]\!]$, t_1 and $t_2 \in [\mathfrak{S}]$, $a \in \mathfrak{A}$, and T is maximally consistent then

if $t_1a{\approx}t_1a \in T$ and $t_1{\approx}t_2 \in T$ then $t_1a{\approx}t_2a \in T$.

Proof

$t_1a{\approx}t_1a \in T$ and $t_1{\approx}t_2 \in T$

$\Rightarrow$ 1	$(t_1a{\approx}t_1a \rightarrow (t_1{\approx}t_2 \rightarrow t_1a{\approx}t_2a))$	A13
2	$t_1a{\approx}t_1a$	T
3	$(t_1{\approx}t_2 \rightarrow t_1a{\approx}t_2a)$	$(1,2)$
4	$t_1{\approx}t_2$	T
5	$t_1a{\approx}t_2a$	$(3,4)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash t_1a{\approx}t_2a$

$\Rightarrow t_1a{\approx}t_2a \in T$. by Lemma 1

□

Lemma 9

If $T \subseteq [\![\mathfrak{S}]\!]$, $t \in [\mathfrak{S}]$, $a \in \mathfrak{A}$, and T is maximally consistent then

if $ta{\approx}ta \in T$ then $t{\approx}t \in T$.

Proof

$ta{\approx}ta \in T$

$\Rightarrow$	1	$(ta{\approx}ta{\rightarrow}t{\approx}t)$	A14
	2	$ta{\approx}ta$	T
	3	$t{\approx}t$	$(1, 2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash t{\approx}t$

$\Rightarrow t{\approx}t \in T$. by Lemma 1

□

Lemma 10

If $T \subseteq [\![\mathfrak{S}]\!]$, and T is maximally consistent then

$\top \in T$.

Proof

1	$\top$	A15

is an $\mathfrak{S}$-derivation from T.

Thus,

$T \vdash \top$.

Thus,

$\top \in T$. by Lemma 1

□

Lemma 11

If $T \subseteq [\![\mathfrak{S}]\!]$, $t \in [\mathfrak{S}]$, $v \in \mathfrak{V}$, and T is maximally consistent then

if $t{\sim}v \in T$ then $t{\approx}t \in T$.

Proof

$t{\sim}v \in T$

$\Rightarrow$	1	$(t{\sim}v{\rightarrow}t{\approx}t)$	A16
	2	$t{\sim}v$	T
	3	$t{\approx}t$	$(1, 2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash t{\approx}t$

$\Rightarrow t{\approx}t \in T$. by Lemma 1

□

Lemma 12

If $T \subseteq [\![\mathfrak{S}]\!]$, $t \in [\mathfrak{S}]$, and T is maximally consistent then

if $t{\approx}t \in T$ then for some $v \in \mathfrak{V}$, $t{\sim}v \in T$.

Proof

$t{\approx}t \in T$

$\Rightarrow$	1	$(t{\approx}t \rightarrow \bigvee\{t{\sim}v \mid v \in \mathfrak{V}\})$	A17
	2	$t{\approx}t$	T
	3	$\bigvee\{t{\sim}v \mid v \in \mathfrak{V}\}$	$(1, 2)$

$\Rightarrow T \vdash \bigvee\{t{\sim}v \mid v \in \mathfrak{V}\}$

$\Rightarrow \bigvee\{t{\sim}v \mid v \in \mathfrak{V}\} \in T$ — by Lemma 1

$\Rightarrow \bot \in T$ or for some $v \in \mathfrak{V}$, $t{\sim}v \in T$ — by Lemma 4

$\Rightarrow$ for some $v \in \mathfrak{V}$, $t{\sim}v \in T$. — since T is consistent

□

Lemma 13

If $T \subseteq [\![\mathfrak{S}]\!]$, $t \in [\mathfrak{S}]$, v_1 and $v_2 \in \mathfrak{V}$, and T is maximally consistent then

if $t{\sim}v_1 \in T$ and $t{\sim}v_2 \in T$ then $v_1 = v_2$.

Proof

$t{\sim}v_1 \in T$, $t{\sim}v_2 \in T$ and $v_1 \neq v_2$

$\Rightarrow$	1	$(t{\sim}v_1 \rightarrow (t{\sim}v_2 \rightarrow \bot))$	A18
	2	$t{\sim}v_1$	T
	3	$(t{\sim}v_2 \rightarrow \bot)$	$(1, 2)$
	4	$t{\sim}v_2$	T
	5	$\bot$	$(3, 4)$

is an $\mathfrak{S}$-derivation form T

$\Rightarrow T \vdash \bot$

$\Rightarrow T$ is inconsistent

$\Rightarrow$ contradiction.

Thus

if $t{\sim}v_1 \in T$ and $t{\sim}v_2 \in T$ then $v_1 = v_2$.

□

Lemma 14

If $T \subseteq [\![\mathfrak{S}]\!]$, t_1 and $t_2 \in [\mathfrak{S}]$, $v \in \mathfrak{V}$, and T is maximally consistent then

if $t_1{\approx}t_2 \in T$ and $t_1{\sim}v \in T$ then $t_2{\sim}v \in T$.

Proof

$t_1{\approx}t_2$ and $t_1{\sim}v \in T$

$\Rightarrow$	1	$(t_1{\approx}t_2{\rightarrow}(t_1{\sim}v{\rightarrow}t_2{\sim}v))$	A19
	2	$t_1{\approx}t_2$	T
	3	$(t_1{\sim}v{\rightarrow}t_2{\sim}v)$	$(1,2)$
	4	$t_1{\sim}v$	T
	5	$t_2{\sim}v$	$(3,4)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash t_2{\sim}v$

$\Rightarrow t_2{\sim}v \in T$. by Lemma 1

□

Proposition 15

If $T \subseteq [\![\mathfrak{S}]\!]$, and T is maximal consistent then T is satisfiable.

Proof

I prove this proposition in two stages. I first construct a structure $S = \langle U, V, A\rangle$ out of the terms in T, then I construct an object $u \in U$ such that $u \in [\![T]\!]_S$.

Stage 1: constructing a structure. For each t_1 and $t_2 \in [\mathfrak{S}]$, let

$t_1 \equiv t_2$ iff $t_1{\approx}t_2 \in T$.

By lemmata 6 and 7, $\equiv$ is an equivalence relation. Let

$D = \{t \in [\mathfrak{S}] \mid t \equiv t\}$.

For each $t \in D$, let

$|t|_\equiv = \{t' \in D \mid t \equiv t'\}$.

Let

$U = \{|t|_\equiv \mid t \in D\}$.

By Lemmata 12, 13 and 14, I can define $V : U \rightarrow \mathfrak{V}$ where

$V(|t|_\equiv) = v$ iff $t{\sim}v \in T$.

By Lemmata 8 and 9, I can define $A : \mathfrak{A} \rightarrow \{U \rightharpoonup U\}$ where

$A(a)$ is defined on $|t|_\equiv$ iff $ta \in D$, and

if $A(a)$ is defined on $|t|_\equiv$ then $A(a)(|t|_\equiv) = |ta|_\equiv$.

Let $S = \langle U, V, A\rangle$.

Claim 1

S is an $\mathfrak{S}$-structure.

Proof of Claim

To show that S meets condition S1 of a structure. For each $a \in \mathfrak{A}$ and $t \in D$,

$A(a)$ *is defined on* $|t|_\equiv$ *and* $V(A(a)(|t|_\equiv)) \notin \mathfrak{S}\langle V(|t|_\equiv), a\rangle$

$\Rightarrow$ $ta \in D$ *and* $V(|ta|_\equiv) \notin \mathfrak{S}\langle V(|t|_\equiv), a\rangle$

$\Rightarrow$ 1	$(t{\sim}V(\lvert t\rvert_\equiv)\rightarrow(ta{\sim}V(\lvert ta\rvert_\equiv)\rightarrow\perp))$	*A20*
2	$t{\sim}V(\lvert t\rvert_\equiv)$	T
3	$(ta{\sim}V(\lvert ta\rvert_\equiv)\rightarrow\perp)$	$(1,2)$
4	$ta{\sim}V(\lvert ta\rvert_\equiv)$	T
5	$\perp$	$(3,4)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash \perp$

$\Rightarrow$ *T is inconsistent*

$\Rightarrow$ *contradiction.*

Thus, for each $a \in \mathfrak{A}$ and $u \in U$

if $A(a)$ is defined on u then $V(A(a)(u)) \in \mathfrak{S}\langle V(u), a\rangle$.

To show that S meets condition S2 of a structure. For each $a \in \mathfrak{A}$ and $t \in D$,

$\mathfrak{S}\langle V(|t|_\equiv), a\rangle \neq \emptyset$

$\Rightarrow$ 1	$(t{\sim}V(\lvert t\rvert_\equiv)\rightarrow ta{\approx}ta)$	*A21*
2	$t{\sim}V(\lvert t\rvert_\equiv)$	T
3	$ta{\approx}ta$	$(1,2)$

is an $\mathfrak{S}$-derivation from T

$\Rightarrow T \vdash ta{\approx}ta$

$\Rightarrow ta{\approx}ta \in T$ *by Lemma 1*

$\Rightarrow ta \equiv ta$

$\Rightarrow ta \in D$

$\Rightarrow$ *$A(a)$ is defined on $|t|_\equiv$.*

Thus, for each $a \in \mathfrak{A}$ and $u \in U$

if $\mathfrak{S}\langle V(u), a\rangle \neq \emptyset$ then $A(a)$ is defined on u.

$\diamond$

Stage 2: constructing an object. By Lemma 10, $: \in D$. Let $u = |:|_{\equiv}$.

Claim 2

If $t \in [\mathfrak{S}]$ then

$[t]_S$ is defined on u iff $t \in D$, and

if $[t]_S$ is defined on u then $[t]_S(u) = |t|_{\equiv}$.

Proof of Claim

By (weak) induction on the length of t.

Start. By Lemma 10, Claim 2 clearly holds when t is $:$.

Step. Suppose that Claim 2 holds for t and that $a \in \mathfrak{A}$. Then

$[ta]_S$ *is defined on* u

$\Leftrightarrow$ $[t]_S$ *is defined on* u *and* $A(a)$ *is defined on* $[t]_S(u)$

$\Leftrightarrow$ $t \in D$ *and* $A(a)$ *is defined on* $|t|_{\equiv}$ — *by the inductive hypothesis*

$\Leftrightarrow$ $ta \in D$,

and

$[ta]_S$ *is defined on* u

$\Rightarrow$ $[t]_S$ *is defined on* u, $A(a)$ *is defined on* $[t]_S(u)$, *and* $[ta]_S(u) = A(a)([t]_S(u))$

$\Rightarrow$ $A(a)$ *is defined on* $|t|_{\equiv}$, *and* $[ta]_S(u) = A(a)(|t|_{\equiv})$ — *by the inductive hypothesis*

$\Rightarrow$ $[ta]_S(u) = |ta|_{\equiv}$.

$\diamond$

Claim 3

If $d \in [\![\mathfrak{S}]\!]$ then

$d \in T$ iff $u \in [\![d]\!]_S$.

Proof of Claim

By (strong) induction on the length of d.

Case 1.

$t{\sim}v \in T$

$\Leftrightarrow$ $t{\approx}t \in T$ *and* $t{\sim}v \in T$ — *by Lemma 11*

$\Leftrightarrow$ $t \equiv t$ and $t{\sim}v \in T$

$\Leftrightarrow$ $t \in D$ *and* $V(|t|_{\equiv}) = v$

$\Leftrightarrow$ $[t]_S$ *is defined on* u *and* $V([t]_S(u)) = v$ — *by Claim 2*

$\Leftrightarrow$ $u \in [\![t{\sim}v]\!]_S$.

Case 2.

$t_1{\approx}t_2 \in T$

$\Leftrightarrow t_1 \equiv t_2$

$\Leftrightarrow t_1 \equiv t_1$, $t_2 \equiv t_2$, and $t_1 \equiv t_2$

$\Leftrightarrow t_1 \in D$, $t_2 \in D$, and $|t_1|_\equiv = |t_2|_\equiv$

$\Leftrightarrow [t_1]_S$ *is defined on* u, $[t_2]_S$ *is defined on* u, and $[t_1]_S(u) = [t_2]_S(u)$ *by Claim 2*

$\Leftrightarrow u \in [\![t_1{\approx}t_2]\!]_S$.

Case 3.

$\neg d \in T$

$\Leftrightarrow d \notin T$ *by Lemma 2*

$\Leftrightarrow u \notin [\![d]\!]_S$ *by the inductive hypothesis*

$\Leftrightarrow u \in U \setminus [\![d]\!]_S$

$\Leftrightarrow u \in [\![\neg d]\!]_S$.

Case 4.

$(d_1{\wedge}d_2) \in T$

$\Leftrightarrow d_1 \in T$ and $d_2 \in T$ *by Lemma 3*

$\Leftrightarrow u \in [\![d_1]\!]_S$ and $u \in [\![d_2]\!]_S$ *by the inductive hypothesis*

$\Leftrightarrow u \in [\![d_1]\!]_S \cap [\![d_2]\!]_S$

$\Leftrightarrow u \in [\![(d_1{\wedge}d_2)]\!]_S$.

Case 5.

$(d_1{\vee}d_2) \in T$

$\Leftrightarrow d_1 \in T$ or $d_2 \in T$ *by Lemma 4*

$\Leftrightarrow u \in [\![d_1]\!]_S$ or $u \in [\![d_2]\!]_S$ *by the inductive hypothesis*

$\Leftrightarrow u \in [\![d_1]\!]_S \cup [\![d_2]\!]_S$

$\Leftrightarrow u \in [\![(d_1{\vee}d_2)]\!]_S$.

Case 6.

$(d_1{\rightarrow}d_2) \in T$

$\Leftrightarrow d_1 \notin T$ or $d_2 \in T$ *by Lemma 5*

$\Leftrightarrow u \notin [\![d_1]\!]_S$ or $u \in [\![d_2]\!]_S$ *by the inductive hypothesis*

$\Leftrightarrow u \in (U \setminus [\![d_1]\!]_S) \cup [\![d_2]\!]_S$

$\Leftrightarrow u \in [\![(d_1{\rightarrow}d_2)]\!]_S$.

$\diamond$

By Claim 3, $u \in [\![T]\!]_S$. Thus, T is satisfiable. $\square$

Proposition 16

If $T \subseteq [\![\mathfrak{S}]\!]$, and T is consistent then T is satisfiable.

Proof

T is consistent

$\Rightarrow$ for some maximally consistent $T' \subseteq [\![\mathfrak{S}]\!]$, $T \subseteq T'$ by Lindenbaum's Lemma

$\Rightarrow$ for some satisfiable $T' \subseteq [\![\mathfrak{S}]\!]$, $T \subseteq T'$ by Proposition 15

$\Rightarrow$ T is satisfiable.

$\square$

Theorem 17 (The Completeness Theorem)

If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$, and $T \models d$ then $T \vdash d$.

Proof

$T \models d$

$\Rightarrow$ $T \cup \{\neg d\}$ is unsatisfiable by Proposition 2 of "Entailment"

$\Rightarrow$ $T \cup \{\neg d\}$ is inconsistent by Proposition 16

$\Rightarrow$ $T \vdash d$. by Proposition 5 of "Derivation"

$\square$

Exhaustive Models

Throughout this handout, fix signature $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ and let ε be the empty string.

Exhaustive Models

If $T \subseteq [\![\mathfrak{S}]\!]$, and S is a structure then S is an exhaustive model of T iff

S is a model of T, and

if $T' \subseteq [\![\mathfrak{S}]\!]$ and $[\![T']\!]_{S'} \neq \emptyset$ for some model S' of T then $[\![T']\!]_S \neq \emptyset$.

Indiscernible Objects

If $S_1 = \langle U_1, V_1, A_1 \rangle$ and $S_2 = \langle U_2, V_2, A_2 \rangle$ are structures then $u_1 \in U_1$ and $u_2 \in U_2$ are indiscernible iff for each $d \in [\![\mathfrak{S}]\!]$,

$u_1 \in [\![d]\!]_{S_1}$ iff $u_2 \in [\![d]\!]_{S_2}$.

Proposition 1

If $S_1 = \langle U_1, V_1, A_1 \rangle$ and $S_2 = \langle U_2, V_2, A_2 \rangle$ are structures then $u_1 \in U_1$ and $u_2 \in U_2$ are indiscernible iff for each $T \subseteq [\![\mathfrak{S}]\!]$,

$u_1 \in [\![T]\!]_{S_1}$ iff $u_2 \in [\![T]\!]_{S_2}$.

Indiscernible Structures

$S_1 = \langle U_1, V_1, A_1 \rangle$ and $S_2 = \langle U_2, V_2, A_2 \rangle$ are indiscernible structures iff

if $u_1 \in U_1$ then for some $u_2 \in U_2$, u_1 and u_2 are indiscernible, and

if $u_2 \in U_2$ then for some $u_1 \in U_1$, u_2 and u_1 are indiscernible.

Proposition 2

If $T \subseteq [\![\mathfrak{S}]\!]$, S_1 is an exhaustive model of T, and S_2 is a model of T then

S_2 is an exhaustive model of T iff S_1 and S_2 are indiscernible.

Proof

Let

$S_1 = \langle U_1, V_1, A_1 \rangle$, and

$S_2 = \langle U_2, V_2, A_2 \rangle$.

For each $u_1 \in U_1$ and $u_2 \in U_2$, let

$T_1(u_1) = \{d \in [\![\mathfrak{S}]\!] \mid u_1 \in [\![d]\!]_{S_1}\}$, and

$T_2(u_2) = \{d \in [\![\mathfrak{S}]\!] \mid u_2 \in [\![d]\!]_{S_2}\}$.

From left to right.

$u_1 \in U_1$

$\Rightarrow u_1 \in [\![T_1(u_1)]\!]_{S_1}$

$\Rightarrow [\![T_1(u_1)]\!]_{S_1} \neq \emptyset$

$\Rightarrow [\![T_1(u_1)]\!]_{S_2} \neq \emptyset$ since S_2 is an exhaustive model of T

$\Rightarrow$ for some $u_2 \in U_2$, $u_2 \in [\![T_1(u_1)]\!]_{S_2}$

$\Rightarrow$ for some $u_2 \in U_2$ and each $d \in [\![\mathfrak{S}]\!]$,

$(u_1 \in [\![d]\!]_{S_1}$

$\Rightarrow d \in T_1(u_1)$

$\Rightarrow u_2 \in [\![d]\!]_{S_2})$

and

$(u_1 \notin [\![d]\!]_{S_1}$

$\Rightarrow u_1 \in [\![\neg d]\!]_{S_1}$

$\Rightarrow \neg d \in T_1(u_1)$

$\Rightarrow u_2 \in [\![\neg d]\!]_{S_2}$

$\Rightarrow u_2 \notin [\![d]\!]_{S_2})$

$\Rightarrow$ for some $u_2 \in U_2$ and each $d \in [\![\mathfrak{S}]\!]$, $u_1 \in [\![d]\!]_{S_1}$ iff $u_2 \in [\![d]\!]_{S_2}$

$\Rightarrow$ for some $u_2 \in U_2$, u_1 and u_2 are indiscernible.

Similarly,

$u_2 \in U_2$

$\Rightarrow$ for some $u_1 \in U_1$, u_2 and u_1 are indiscernible.

From right to left. If $T' \subseteq [\![\mathfrak{S}]\!]$ then

$[\![T']\!]_{S'} \neq \emptyset$ for some model S' of T

$\Rightarrow [\![T']\!]_{S_1} \neq \emptyset$

$\Rightarrow$ for some $u_1 \in U_1$, $u_1 \in [\![T']\!]_{S_1}$

$\Rightarrow$ for some $u_1 \in U_1$ and some $u_2 \in U_2$,
$u_1 \in [\![T']\!]_{S_1}$ and u_1 and u_2 are indiscernible
since S_1 and S_2 are indiscernible

$\Rightarrow$ for some $u_2 \in U_2$, $u_2 \in [\![T']\!]_{S_2}$ by Proposition 1

$\Rightarrow [\![T']\!]_{S_2} \neq \emptyset$.

$\square$

Morphs

A morph is a pair $\langle g, l\rangle$ where $g \subseteq \mathfrak{S} \times \mathfrak{S}$ and $l : \mathfrak{S} \rightharpoonup \mathfrak{V}$ such that

if $\langle s_1, s_2\rangle \in g$ then $\langle s_2, s_1\rangle \in g$,

if $\langle s_1, s_2\rangle \in g$ and $\langle s_2, s_3\rangle \in g$ then $\langle s_1, s_3\rangle \in g$,

if $a \in \mathfrak{A}$, $\langle s_1 a, s_1 a\rangle \in g$ and $\langle s_1, s_2\rangle \in g$ then $\langle s_1 a, s_2 a\rangle \in g$,

if $a \in \mathfrak{A}$ and $\langle sa, sa\rangle \in g$ then $\langle s, s\rangle \in g$,

$\langle \varepsilon, \varepsilon\rangle \in g$,

$\langle s, s\rangle \in g$ iff l is defined on s,

if $\langle s_1, s_2\rangle \in g$ then $l(s_1) = l(s_2)$,

if $a \in \mathfrak{A}$ and $\langle sa, sa\rangle \in g$ then $l(sa) \in \mathfrak{S}\langle l(s), a\rangle$, and

if $a \in \mathfrak{A}$, $\langle s, s\rangle \in g$ and $\mathfrak{S}\langle l(s), a\rangle \neq \emptyset$ then $\langle sa, sa\rangle \in g$.

Write $\mathfrak{M}$ for the set of morphs.

If $\langle g, l\rangle \in \mathfrak{M}$ then g is an equivalence relation. For each $\langle g, l\rangle \in \mathfrak{M}$, let

$Dom(g) = \{s \in \mathfrak{S} \mid \langle s, s\rangle \in g\}$,

and for each $s \in Dom(g)$, let

$|s|_g = \{s' \in \mathfrak{S} \mid \langle s, s'\rangle \in g\}$.

Morphologies

If $S = \langle U, V, A\rangle$ is a structure then the S-morphology is the function $\{\![\ ]\!\}_S : U \to \mathfrak{M}$ where $\{\![u]\!\}_S = \langle g, l\rangle$ iff

$g = \{\langle s_1, s_2\rangle \in \mathfrak{S} \times \mathfrak{S} \mid u \in [\![{:}s_1{\approx}{:}s_2]\!]_S\}$, and

$l = \{\langle s, v\rangle \in \mathfrak{S} \times \mathfrak{V} \mid u \in [\![{:}s{\sim}v]\!]_S\}$.

Submorphs

If $\langle g, l\rangle \in \mathfrak{M}$ and $s \in \mathfrak{S}$ then

$g/s = \{\langle s_1, s_2\rangle \mid \langle ss_1, ss_2\rangle \in g\}$,

$l/s = \{\langle s', v\rangle \mid \langle ss', v\rangle \in l\}$, and

$\langle g, l\rangle/s = \langle g/s, l/s\rangle$.

Proposition 3

If $\langle g, l\rangle \in \mathfrak{M}$ and $s \in Dom(g)$ then $\langle g, l\rangle/s \in \mathfrak{M}$.

If $\langle g, l\rangle \in \mathfrak{M}$ and $s \in Dom(g)$ then say s generates submorph $\langle g, l\rangle/s$ of $\langle g, l\rangle$.

Proposition 4

If $S = \langle U, V, A\rangle$ *is a structure,* $u \in U$*,* $s \in \mathfrak{S}$*, and* $[{:}s]_S$ *is defined on* u *then*

$$\{\![\, [{:}s]_S(u)]\!\}_S = \{\![u]\!\}_S / s.$$

Proof

Suppose

$[{:}s]_S(u) = u'$,

$\{\![u]\!\}_S = \langle g, l\rangle$, and

$\{\![u']\!\}_S = \langle g', l'\rangle$.

Then

$\langle s_1, s_2\rangle \in g'$

$\Leftrightarrow u' \in [\![{:}s_1 \approx {:}s_2]\!]_S$

$\Leftrightarrow [{:}s_1]_S$ is defined on u',
$[{:}s_2]_S$ is defined on u', and
$[{:}s_1]_S(u') = [{:}s_2]_S(u')$

$\Leftrightarrow [{:}ss_1]_S$ is defined on u,
$[{:}ss_2]_S$ is defined on u, and
$[{:}ss_1]_S(u) = [{:}ss_2]_S(u)$

$\Leftrightarrow u \in [\![{:}ss_1 \approx {:}ss_2]\!]_S$

$\Leftrightarrow \langle ss_1, ss_2\rangle \in g$

$\Leftrightarrow \langle s_1, s_2\rangle \in g/s$,

and

$\langle s', v\rangle \in l'$

$\Leftrightarrow u' \in [\![{:}s' \sim v]\!]_S$

$\Leftrightarrow [{:}s']_S$ is defined on u', and
$V([{:}s']_S(u')) = v$

$\Leftrightarrow [{:}ss']_S$ is defined on u, and
$V([{:}ss']_S(u)) = v$

$\Leftrightarrow u \in [\![{:}ss' \sim v]\!]_S$

$\Leftrightarrow \langle ss', v\rangle \in l$

$\Leftrightarrow \langle s', v\rangle \in l/s$.

□

Envelopes

The (description) envelope is the function $\mathfrak{E} : [\![\mathfrak{S}]\!] \to Pow(\mathfrak{M})$ where

$\mathfrak{E}(:s{\sim}v) = \{\langle g, l\rangle \in \mathfrak{M} \mid \langle s, v\rangle \in l\}$,

$\mathfrak{E}(:s_1{\approx}:s_2) = \{\langle g, l\rangle \in \mathfrak{M} \mid \langle s_1, s_2\rangle \in g\}$,

$\mathfrak{E}(\neg d) = \mathfrak{M} \setminus \mathfrak{E}(d)$,

$\mathfrak{E}((d_1{\wedge}d_2)) = \mathfrak{E}(d_1) \cap \mathfrak{E}(d_2)$,

$\mathfrak{E}((d_1{\vee}d_2)) = \mathfrak{E}(d_1) \cup \mathfrak{E}(d_2)$, and

$\mathfrak{E}((d_1{\to}d_2)) = (\mathfrak{M} \setminus \mathfrak{E}(d_1)) \cup \mathfrak{E}(d_2)$.

Proposition 5

If $d \in [\![\mathfrak{S}]\!]$, $S = \langle U, V, A\rangle$ *is a structure, and* $u \in U$ *then*

$u \in [\![d]\!]_S$ *iff* $\{\![u]\!\}_S \in \mathfrak{E}(d)$.

Proof

By an easy (strong) induction on the length of d. $\square$

The (theory) envelope is the function $\mathfrak{E} : Pow([\![\mathfrak{S}]\!]) \to Pow(\mathfrak{M})$ where

$\mathfrak{E}(T) = \bigcap\{\mathfrak{E}(d) \mid d \in T\}$.

Proposition 6

If $T \subseteq [\![\mathfrak{S}]\!]$, $S = \langle U, V, A\rangle$ *is a structure, and* $u \in U$ *then*

$u \in [\![T]\!]_S$ *iff* $\{\![u]\!\}_S \in \mathfrak{E}(T)$.

Characters

The character is the function $\mathfrak{C} : Pow([\![\mathfrak{S}]\!]) \to Pow(\mathfrak{M})$ where

$\mathfrak{C}(T) = \{\langle g, l\rangle \in \mathfrak{M} \mid \langle g, l\rangle/s \in \mathfrak{E}(T)$ for each $s \in Dom(g)\}$.

Proposition 7

If $T \subseteq [\![\mathfrak{S}]\!]$, $S = \langle U, V, A\rangle$ *is a model of* T, *and* $u \in U$ *then*

$\{\![u]\!\}_S \in \mathfrak{C}(T)$.

Proof

Suppose

$\{\![u]\!\}_S = \langle g, l\rangle$.

Then for each $s \in \mathfrak{S}$,

$s \in Dom(g)$

$\Rightarrow \langle s, s\rangle \in g$

$\Rightarrow u \in [\![:s{\approx}:s]\!]_S$

$\Rightarrow [{:}s]_S$ is defined on u

$\Rightarrow \{\![\, [{:}s]_S(u)]\!\}_S = \langle g, l\rangle/s$ and $[{:}s]_S(u) \in U$ by Proposition 4

$\Rightarrow \{\![\, [{:}s]_S(u)]\!\}_S = \langle g, l\rangle/s$ and $[{:}s]_S(u) \in [\![T]\!]_S$ since S is a model of T

$\Rightarrow \{\![\, [{:}s]_S(u)]\!\}_S = \langle g, l\rangle/s$ and $\{\![\, [{:}s]_S(u)]\!\}_S \in \mathfrak{E}(T)$ by Proposition 6

$\Rightarrow \langle g, l\rangle/s \in \mathfrak{E}(T)$.

□

Feature Structures

A feature structure is a triple $\langle g, l, c\rangle$ where $\langle g, l\rangle \in \mathfrak{M}$ and c is an equivalence class of g. Write $\mathfrak{F}$ for the set of feature structures.

If $T \subseteq [\![\mathfrak{S}]\!]$ then

$U_T = \{\langle g, l, |s|_g\rangle \in \mathfrak{F} \mid \langle g, l\rangle \in \mathfrak{E}(T)\}$;

$V_T : U_T \to \mathfrak{V}$ by $V_T\langle g, l, |s|_g\rangle = l(s)$;

$A_T : \mathfrak{A} \to \{U_T \rightharpoonup U_T\}$ by

$A_T(a)$ is defined on $\langle g, l, |s|_g\rangle$ iff $sa \in Dom(g)$, and

if $A_T(a)$ is defined on $\langle g, l, |s|_g\rangle$
then $A_T(a)\langle g, l, |s|_g\rangle = \langle g, l, |sa|_g\rangle$; and

$S_T = \langle U_T, V_T, A_T\rangle$.

Proposition 8

If $T \subseteq [\![\mathfrak{S}]\!]$ then S_T is a structure.

Proof

To show that S_T meets condition S1. For each $a \in \mathfrak{A}$ and $\langle g, l, |s|_g\rangle \in U_T$,

$A_T(a)$ is defined on $\langle g, l, |s|_g\rangle$

$\Rightarrow sa \in Dom(g)$ and $A_T(a)\langle g, l, |s|_g\rangle = \langle g, l, |sa|_g\rangle$

$\Rightarrow l(sa) \in \mathfrak{S}\langle l(s), a\rangle$ and $V_T(A_T(a)\langle g, l, |s|_g\rangle) = l(sa)$

$\Rightarrow V_T(A_T(a)\langle g, l, |s|_g\rangle) \in \mathfrak{S}\langle V_T\langle g, l, |s|_g\rangle, a\rangle$.

To show that S_T meets condition S2. For each $a \in \mathfrak{A}$ and $\langle g, l, |s|_g\rangle \in U_T$,

$\mathfrak{S}\langle V_T\langle g, l, |s|_g\rangle, a\rangle \neq \emptyset$

$\Rightarrow \mathfrak{S}\langle l(s), a\rangle \neq \emptyset$

$\Rightarrow sa \in Dom(g)$

$\Rightarrow A_T(a)$ is defined on $\langle g, l, |s|_g\rangle$.

□

Proposition 9

If $T \subseteq [\![\mathfrak{S}]\!]$, *and* $\langle g, l, |s|_g \rangle \in U_T$ *then*

$$\{\![\langle g, l, |s|_g \rangle]\!\}_{S_T} = \langle g, l \rangle / s.$$

Proof

Suppose

$$\{\![\langle g, l, |s|_g \rangle]\!\}_{S_T} = \langle g', l' \rangle.$$

Then

$$\begin{aligned}
&\langle s_1, s_2 \rangle \in g' \\
&\Leftrightarrow \langle g, l, |s|_g \rangle \in [\![:s_1 \approx :s_2]\!]_{S_T} \\
&\Leftrightarrow [:s_1]_{S_T} \text{ is defined on } \langle g, l, |s|_g \rangle, \\
&\quad\ [:s_2]_{S_T} \text{ is defined on } \langle g, l, |s|_g \rangle, \text{ and} \\
&\quad\ [:s_1]_{S_T} \langle g, l, |s|_g \rangle = [:s_2]_{S_T} \langle g, l, |s|_g \rangle \\
&\Leftrightarrow ss_1 \in Dom(g), \\
&\quad\ ss_2 \in Dom(g), \text{ and} \\
&\quad\ \langle g, l, |ss_1|_g \rangle = \langle g, l, |ss_2|_g \rangle \\
&\Leftrightarrow \langle ss_1, ss_2 \rangle \in g \\
&\Leftrightarrow \langle s_1, s_2 \rangle \in g/s,
\end{aligned}$$

and

$$\begin{aligned}
&\langle s', v \rangle \in l' \\
&\Leftrightarrow \langle g, l, |s|_g \rangle \in [\![:s' \sim v]\!]_{S_T} \\
&\Leftrightarrow [:s']_{S_T} \text{ is defined on } \langle g, l, |s|_g \rangle, \text{ and} \\
&\quad\ V_T([:s']_{S_T} \langle g, l, |s|_g \rangle) = v \\
&\Leftrightarrow ss' \in Dom(g), \text{ and} \\
&\quad\ V_T(\langle g, l, |ss'|_g \rangle) = v \\
&\Leftrightarrow \langle ss', v \rangle \in l \\
&\Leftrightarrow \langle s', v \rangle \in l/s.
\end{aligned}$$

□

Proposition 10

If $T \subseteq [\![\mathfrak{S}]\!]$ *then* S_T *is a model of* T.

Proof

By Proposition 8, S_T is a structure. Further,

$$\begin{aligned}
&\langle g, l, |s|_g \rangle \in U_T \\
&\Rightarrow \langle g, l \rangle \in \mathfrak{C}(T) \text{ and } s \in Dom(g) \\
&\Rightarrow \langle g, l \rangle / s \in \mathfrak{C}(T)
\end{aligned}$$

$\Rightarrow \{\![\langle g, l, |s|_g\rangle]\!\}_{S_T} \in \mathfrak{E}(T)$ by Proposition 9

$\Rightarrow \langle g, l, |s|_g\rangle \in [\![T]\!]_{S_T}$. by Proposition 6

□

Proposition 11

If $T \subseteq [\![\mathfrak{S}]\!]$ then S_T is an exhaustive model of T.

Proof

By Proposition 10, S_T is a model of T. Further, for each $T' \subseteq [\![\mathfrak{S}]\!]$,

$[\![T']\!]_S \neq \emptyset$ for some model S of T

$\Rightarrow$ for some model $S = \langle U, V, A\rangle$ of T and some $u \in U$,
$\{\![u]\!\}_S \in \mathfrak{E}(T)$ and $u \in [\![T']\!]_S$ by Proposition 7

$\Rightarrow$ for some model $S = \langle U, V, A\rangle$ of T and some $u \in U$,
$\{\![u]\!\}_S \in \mathfrak{E}(T)$ and $\{\![u]\!\}_S \in \mathfrak{E}(T')$ by Proposition 6

$\Rightarrow$ for some $\langle g, l\rangle \in \mathfrak{M}$, $\langle g, l\rangle \in \mathfrak{E}(T)$ and $\langle g, l\rangle \in \mathfrak{E}(T')$

$\Rightarrow$ for some $\langle g, l\rangle \in \mathfrak{M}$, $\langle g, l, |\varepsilon|_g\rangle \in U_T$ and $\{\![\langle g, l, |\varepsilon|_g\rangle]\!\}_{S_T} \in \mathfrak{E}(T')$
by Proposition 9

$\Rightarrow$ for some $u \in U_T$, $u \in [\![T']\!]_{S_T}$ by Proposition 6

$\Rightarrow [\![T']\!]_{S_T} \neq \emptyset$

□

Morphs and Maximally Consistent Theories

Fix a signature $\mathfrak{S} : \mathfrak{V} \times \mathfrak{A} \to Pow(\mathfrak{V})$ throughout.

From Maximally Consistent Theories to Morphs

Let

$Max = \{T \subseteq [\![\mathfrak{S}]\!] \mid T \text{ is maximally consistent}\}$,

and let $\mu : Max \to \mathfrak{M}$ where $\mu(T) = \langle g, l\rangle$ iff

$g = \{\langle s_1, s_2\rangle \mid {:}s_1{\approx}{:}s_2 \in T\}$, and

$l = \{\langle s, v\rangle \mid {:}s{\sim}v \in T\}$.

Proposition 1

μ is well defined.

Proof

Let $T \in Max$ and let $\mu(T) = \langle g, l\rangle$. I must show that $\langle g, l\rangle \in \mathfrak{M}$. l is a function, since

$\langle s, v_1\rangle \in l$ and $\langle s, v_2\rangle \in l$

$\Rightarrow {:}s{\sim}v_1 \in T$ and ${:}s{\sim}v_2 \in T$

$\Rightarrow v_1 = v_2$. by "Completeness" Lemma 13

Additionally $\langle g, l\rangle$ meets the nine conditions for a morph. Firstly,

$\langle s_1, s_2\rangle \in g$

$\Rightarrow {:}s_1{\approx}{:}s_2 \in T$

$\Rightarrow {:}s_2{\approx}{:}s_1 \in T$ by "Completeness" Lemma 6

$\Rightarrow \langle s_2, s_1\rangle \in g$.

Secondly,

$\langle s_1, s_2\rangle \in g$ and $\langle s_2, s_3\rangle \in g$

$\Rightarrow {:}s_1{\approx}{:}s_2 \in T$ and ${:}s_2{\approx}{:}s_3 \in T$

$\Rightarrow {:}s_1{\approx}{:}s_3 \in T$ by "Completeness" Lemma 7

$\Rightarrow \langle s_1, s_3\rangle \in g$.

Thirdly, if $a \in \mathfrak{A}$ then

$\langle s_1a, s_1a\rangle \in g$ and $\langle s_1, s_2\rangle \in g$

$\Rightarrow {:}s_1a{\approx}{:}s_1a \in T$ and ${:}s_1{\approx}{:}s_2 \in T$

$\Rightarrow {:}s_1a{\approx}{:}s_2a \in T$ by "Completeness" Lemma 8

$\Rightarrow \langle s_1a, s_2a\rangle \in g$.

Fourthly, if $a \in \mathfrak{A}$ then

$\langle sa, sa\rangle \in g$

$\Rightarrow$ $:sa{\approx}:sa \in T$

$\Rightarrow$ $:s{\approx}:s \in T$ by "Completeness" Lemma 9

$\Rightarrow$ $\langle s, s\rangle \in g$.

Fifthly,

$:{\approx}: \in T$. by "Completeness" Lemma 10

Thus,

$\langle \varepsilon, \varepsilon\rangle \in g$.

Sixthly,

$\langle s, s\rangle \in g$

$\Leftrightarrow$ $:s{\approx}:s \in T$

$\Leftrightarrow$ for some $v \in \mathfrak{V}$, $:s{\sim}v \in T$ by "Completeness" Lemmata 11 and 12

$\Leftrightarrow$ for some $v \in \mathfrak{V}$, $\langle s, v\rangle \in l$.

Seventhly,

$\langle s_1, s_2\rangle \in g$ and $l(s_1) \neq l(s_2)$

$\Rightarrow$ $:s_1{\approx}:s_2 \in T$, $:s_1{\sim}l(s_1) \in T$, and $:s_2{\sim}l(s_1) \notin T$

$\Rightarrow$ contradiction. by "Completeness" Lemma 14

Thus,

if $\langle s_1, s_2\rangle \in g$ then $l(s_1) = l(s_2)$.

Eighthly, if $a \in \mathfrak{A}$ then

$\langle sa, sa\rangle \in g$ and $l(sa) \notin \mathfrak{S}\langle l(s), a\rangle$

$\Rightarrow$ $:s{\sim}l(s) \in T$, $:sa{\sim}l(sa) \in T$, and $l(sa) \notin \mathfrak{S}\langle l(s), a\rangle$

$\Rightarrow$

1	$(:s{\sim}l(s){\rightarrow}(:sa{\sim}l(sa){\rightarrow}\bot))$	A20
2	$:s{\sim}l(s)$	T
3	$(:sa{\sim}l(sa){\rightarrow}\bot)$	$(1, 2)$
4	$:sa{\sim}l(sa)$	T
5	$\bot$	$(3, 4)$

is a derivation from T

$\Rightarrow$ $T \vdash \bot$

$\Rightarrow$ T is inconsistent

$\Rightarrow$ contradiction.

Thus

$$\text{if } \langle sa, sa\rangle \in g \text{ then } l(sa) \notin \mathfrak{S}\langle l(s), a\rangle.$$

Lastly, if $a \in \mathfrak{A}$ then

$\langle s, s\rangle \in g$ and $\mathfrak{S}\langle l(s), a\rangle \neq \emptyset$

$\Rightarrow :s{\sim}l(s) \in T$ and $\mathfrak{S}\langle l(s), a\rangle \neq \emptyset$

$\Rightarrow$	1	$(:s{\sim}l(s) \rightarrow :sa{\approx}:sa)$	A21
	2	$:s{\sim}l(s)$	T
	3	$:sa{\approx}:sa$	$(1, 2)$

is a derivation form T

$\Rightarrow T \vdash :sa{\approx}:sa$

$\Rightarrow :sa{\approx}:sa \in T$ by "Completeness" Lemma 1

$\Rightarrow \langle sa, sa\rangle \in g$.

□

From Morphs to Maximally Consistent Theories

Clearly,

$$\mathfrak{E}(\emptyset) = \mathfrak{M}.$$

Thus, for each $\langle g, l\rangle \in \mathfrak{M}$,

$s \in Dom(g)$

$\Rightarrow \langle g, l\rangle/s \in \mathfrak{M}$ by "Exhaustive Models" Proposition 3

$\Rightarrow \langle g, l\rangle/s \in \mathfrak{E}(\emptyset)$.

Thus,

$$\mathfrak{C}(\emptyset) = \mathfrak{M},$$

and

$$U_\emptyset = \mathfrak{F}.$$

Therefore,

$\langle g, l\rangle \in \mathfrak{M}$

$\Rightarrow \langle g, l, |\varepsilon|_g\rangle \in \mathfrak{F}$

$\Rightarrow \langle g, l, |\varepsilon|_g\rangle \in U_\emptyset$.

Let $\theta : \mathfrak{M} \rightarrow Max$ where

$$\theta\langle g, l\rangle = \{d \in [\![\mathfrak{S}]\!] \mid \langle g, l, |\varepsilon|_g\rangle \in [\![d]\!]_{S_\emptyset}\}.$$

Proposition 2

θ is well defined.

Proof

Let $\langle g, l\rangle \in \mathfrak{M}$. I must show that $\theta\langle g, l\rangle \in Max$.

$\langle g, l, |\varepsilon|_g\rangle \in [\![\theta\langle g, l\rangle]\!]_{S_\emptyset}$.

Thus,

$\theta\langle g, l\rangle$ is satisfiable.

Thus,

$\theta\langle g, l\rangle$ is consistent. by "Soundness" Corollary 5

Further,

$\theta\langle g, l\rangle \subset T$

$\Rightarrow$ for some $d \in [\![\mathfrak{S}]\!]$, $\theta\langle g, l\rangle \cup \{d\} \subseteq T$ and $d \notin \theta\langle g, l\rangle$

$\Rightarrow$ for some $d \in [\![\mathfrak{S}]\!]$, $\theta\langle g, l\rangle \cup \{d\} \subseteq T$ and $\langle g, l, |\varepsilon|_g\rangle \notin [\![d]\!]_{S_\emptyset}$

$\Rightarrow$ for some $d \in [\![\mathfrak{S}]\!]$, $\theta\langle g, l\rangle \cup \{d\} \subseteq T$ and $\langle g, l, |\varepsilon|_g\rangle \in [\![\neg d]\!]_{S_\emptyset}$

$\Rightarrow$ for some $d \in [\![\mathfrak{S}]\!]$, $\theta\langle g, l\rangle \cup \{d\} \subseteq T$ and $\neg d \in \theta\langle g, l\rangle$

$\Rightarrow$ for some $d \in [\![\mathfrak{S}]\!]$, $d \in T$ and $\neg d \in T$

$\Rightarrow$ for some $d \in [\![\mathfrak{S}]\!]$,

1	$((\neg\bot\to\neg d)\to(d\to\bot))$	A3
2	$(\neg d\to(\neg\bot\to\neg d))$	A1
3	$\neg d$	T
4	$(\neg\bot\to\neg d)$	$(2,3)$
5	$(d\to\bot)$	$(1,4)$
6	d	T
7	$\bot$	$(5,6)$

is a derivation from T

$\Rightarrow T \vdash \bot$

$\Rightarrow T$ is inconsistent.

Thus, $\theta\langle g, l\rangle$ is maximally consistent. □

Establishing Bijections

Lemma 3

If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$, and $T \in Max$ then

$d \in T$ *iff* $d \in \theta(\mu(T))$.

Proof

By (strong) induction on the length of d. Let $\mu(T) = \langle g, l\rangle$.

Case 1.

$:s{\sim}v \in T$

$\Leftrightarrow \langle s, v\rangle \in l$

$\Leftrightarrow s \in \mathit{Dom}(g)$ and $l(s) = v$

$\Leftrightarrow [{:}s]_{S_\emptyset}$ is defined on $\langle g, l, |\varepsilon|_g\rangle$, and $V_\emptyset([{:}s]_{S_\emptyset}\langle g, l, |\varepsilon|_g\rangle) = v$

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![{:}s{\sim}v]\!]_{S_\emptyset}$

$\Leftrightarrow {:}s{\sim}v \in \theta\langle g, l\rangle$.

Case 2.

${:}s_1{\approx}{:}s_2 \in T$

$\Leftrightarrow \langle s_1, s_2\rangle \in g$

$\Leftrightarrow s_1 \in \mathit{Dom}(g)$, $s_2 \in \mathit{Dom}(g)$ and $|s_1|_g = |s_2|_g$

$\Leftrightarrow [{:}s_1]_{S_\emptyset}$ is defined on $\langle g, l, |\varepsilon|_g\rangle$, $[{:}s_2]_{S_\emptyset}$ is defined on $\langle g, l, |\varepsilon|_g\rangle$, and $[{:}s_1]_{S_\emptyset}\langle g, l, |\varepsilon|_g\rangle = [{:}s_2]_{S_\emptyset}\langle g, l, |\varepsilon|_g\rangle$

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![{:}s_1{\approx}{:}s_2]\!]_{S_\emptyset}$

$\Leftrightarrow {:}s_1{\approx}{:}s_2 \in \theta\langle g, l\rangle$.

Case 3.

$\neg d \in T$

$\Leftrightarrow d \notin T$ by "Completeness" Lemma 2

$\Leftrightarrow d \notin \theta(\mu(T))$ by the inductive hypothesis

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \notin [\![d]\!]_{S_\emptyset}$

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![\neg d]\!]_{S_\emptyset}$

$\Leftrightarrow \neg d \in \theta\langle g, l\rangle$.

Case 4.

$(d_1{\wedge}d_2) \in T$

$\Leftrightarrow d_1 \in T$ and $d_2 \in T$ by "Completeness" Lemma 3

$\Leftrightarrow d_1 \in \theta(\mu(T))$ and $d_2 \in \theta(\mu(T))$ by the inductive hypothesis

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![d_1]\!]_{S_\emptyset}$ and $\langle g, l, |\varepsilon|_g\rangle \in [\![d_2]\!]_{S_\emptyset}$

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![(d_1{\wedge}d_2)]\!]_{S_\emptyset}$

$\Leftrightarrow (d_1{\wedge}d_2) \in \theta\langle g, l\rangle$.

Case 5.

$(d_1 \vee d_2) \in T$

$\Leftrightarrow d_1 \in T$ or $d_2 \in T$ by "Completeness" Lemma 4

$\Leftrightarrow d_1 \in \theta(\mu(T))$ or $d_2 \in \theta(\mu(T))$ by the inductive hypothesis

$\Leftrightarrow \langle g, l, |\varepsilon|_g \rangle \in [\![d_1]\!]_{S_\emptyset}$ or $\langle g, l, |\varepsilon|_g \rangle \in [\![d_2]\!]_{S_\emptyset}$

$\Leftrightarrow \langle g, l, |\varepsilon|_g \rangle \in [\![(d_1 \vee d_2)]\!]_{S_\emptyset}$

$\Leftrightarrow (d_1 \vee d_2) \in \theta\langle g, l \rangle$.

Case 6.

$(d_1 \rightarrow d_2) \in T$

$\Leftrightarrow d_1 \notin T$ or $d_2 \in T$ by "Completeness" Lemma 5

$\Leftrightarrow d_1 \notin \theta(\mu(T))$ or $d_2 \in \theta(\mu(T))$ by the inductive hypothesis

$\Leftrightarrow \langle g, l, |\varepsilon|_g \rangle \notin [\![d_1]\!]_{S_\emptyset}$ or $\langle g, l, |\varepsilon|_g \rangle \in [\![d_2]\!]_{S_\emptyset}$

$\Leftrightarrow \langle g, l, |\varepsilon|_g \rangle \in [\![(d_1 \rightarrow d_2)]\!]_{S_\emptyset}$

$\Leftrightarrow (d_1 \rightarrow d_2) \in \theta\langle g, l \rangle$.

$\square$

Proposition 4

If $T \in Max$ then $T = \theta(\mu(T))$.

Proposition 5

If $\langle g, l \rangle \in \mathfrak{M}$ then $\langle g, l \rangle = \mu(\theta\langle g, l \rangle)$.

Proof

Let $\mu(\theta\langle g, l \rangle) = \langle g', l' \rangle$. Then

$\langle s_1, s_2 \rangle \in g$

$\Leftrightarrow s_1 \in Dom(g)$, $s_2 \in Dom(g)$ and $\langle g, l, |s_1|_g \rangle = \langle g, l, |s_2|_g \rangle$

$\Leftrightarrow [{:}s_1]_{S_\emptyset}$ is defined on $\langle g, l, |\varepsilon|_g \rangle$,
$[{:}s_2]_{S_\emptyset}$ is defined on $\langle g, l, |\varepsilon|_g \rangle$, and
$[{:}s_1]_{S_\emptyset}\langle g, l, |\varepsilon|_g \rangle = [{:}s_2]_{S_\emptyset}\langle g, l, |\varepsilon|_g \rangle$

$\Leftrightarrow \langle g, l, |\varepsilon|_g \rangle \in [\![{:}s_1 \approx {:}s_2]\!]_{S_\emptyset}$

$\Leftrightarrow {:}s_1 \approx {:}s_2 \in \theta\langle g, l \rangle$

$\Leftrightarrow \langle s_1, s_2 \rangle \in g'$,

and

$\langle s, v \rangle \in l$

$\Leftrightarrow s \in Dom(g)$ and $V_\emptyset\langle g, l, |s|_g \rangle = v$

$\Leftrightarrow [{:}s]_{S_\emptyset}$ is defined on $\langle g, l, |\varepsilon|_g\rangle$, and
$V_\emptyset([{:}s]_{S_\emptyset}\langle g, l, |\varepsilon|_g\rangle) = v$

$\Leftrightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![{:}s{\sim}v]\!]_{S_\emptyset}$

$\Leftrightarrow {:}s{\sim}v \in \theta\langle g, l\rangle$

$\Leftrightarrow \langle s, v\rangle \in l'$.

□

Thus, μ and θ are bijections and θ is the inverse of μ. Notice that though the definition of θ uses the structure $S_\emptyset$, the definition of μ uses no structure at all. Hence, θ can be redefined without using a structure.

Envelopes of Maximally Consistent Theories

Lemma 6

If $T \cup \{d\} \subseteq [\![\mathfrak{S}]\!]$, and $T \in Max$ then

$d \in T$ *iff* $\mu(T) \in \mathfrak{E}(d)$.

Proof

By an easy (strong) induction on the length of d. □

Proposition 7

If $T \in Max$ then $\mu(T) \in \mathfrak{E}(T)$.

Proposition 8

If $T \in Max$ and $\langle g, l\rangle \in \mathfrak{E}(T)$ then $\langle g, l\rangle = \mu(T)$.

Proof

$\langle g, l\rangle \in \mathfrak{E}(T)$

$\Rightarrow \{\![\langle g, l, |\varepsilon|_g\rangle]\!\}_{S_\emptyset} \in \mathfrak{E}(T)$ by "Exhaustive Models" Proposition 9

$\Rightarrow \langle g, l, |\varepsilon|_g\rangle \in [\![T]\!]_{S_\emptyset}$ by "Exhaustive Models" Proposition 6

$\Rightarrow T \subseteq \theta\langle g, l\rangle$

$\Rightarrow T = \theta\langle g, l\rangle$ since T and $\theta\langle g, l\rangle$ are maximally consistent

$\Rightarrow \mu(T) = \mu(\theta\langle g, l\rangle)$

$\Rightarrow \mu(T) = \langle g, l\rangle$. by Proposition 5

□

Thus, if T is maximally consistent then

$\mathfrak{E}(T) = \{\mu(T)\}$

and for each structure $S = \langle U, V, A\rangle$ and each $u \in U$,

$u \in [\![T]\!]_S$ iff $\{\![u]\!\}_S = \mu(T)$. by "Exhaustive Models" Proposition 6