On the Existence of Exhaustive Models in a Relational Feature Logic for Head-Driven Phrase Structure Grammar

- draft -

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Contents

1 RSRL

1 Definition. Var *is a countably infinite set of symbols.*

2 Definition. Σ *is a signature iff*

Σ *is a septuple* $\langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$,

$\langle \mathcal{G}, \sqsubseteq \rangle$ *is a finite partial order,*

$\mathcal{S} = \left\{ \sigma \in \mathcal{G} \;\middle|\; \begin{array}{l} \text{for each } \sigma' \in \mathcal{G}, \\ \text{if } \sigma' \sqsubseteq \sigma \text{ then } \sigma = \sigma' \end{array} \right\}$,

$\mathcal{A}$ *is a set,*

$\mathcal{F}$ *is a partial function from the Cartesian product of* $\mathcal{G}$ *and* $\mathcal{A}$ *to* $\mathcal{G}$, *and*

> *for each* $\sigma_1 \in \mathcal{G}$, *for each* $\sigma_2 \in \mathcal{G}$ *and for each* $\alpha \in \mathcal{A}$,
>
> > *if* $\mathcal{F}\langle \sigma_1, \alpha \rangle$ *is defined and* $\sigma_2 \sqsubseteq \sigma_1$
> > *then* $\mathcal{F}\langle \sigma_2, \alpha \rangle$ *is defined and* $\mathcal{F}\langle \sigma_2, \alpha \rangle \sqsubseteq \mathcal{F}\langle \sigma_1, \alpha \rangle$,

$\mathcal{R}$ *is a finite set, and*

$\mathcal{AR}$ *is a total function from* $\mathcal{R}$ *to* $\mathbb{N}^+$.

Suppose S is a set. Throughout this paper we write $\overline{S}$ as an abbreviation for $S \cup S^*$.

3 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, I *is a* Σ *interpretation iff*

I *is a quadruple* $\langle U, S, A, R \rangle$,

U *is a set of objects,*

S *is a total function from* U *to* $\mathcal{S}$,

A *is a total function from* $\mathcal{A}$ *to the set of partial functions from* U *to* U,

for each $\alpha \in \mathcal{A}$ *and each* $u \in U$,

> *if* $A(\alpha)(u)$ *is defined*
> *then* $\mathcal{F}\langle S(u), \alpha \rangle$ *is defined, and* $S(A(\alpha)(u)) \sqsubseteq \mathcal{F}\langle S(u), \alpha \rangle$, *and*

for each $\alpha \in \mathcal{A}$ *and each* $u \in U$,

> *if* $\mathcal{F}\langle S(u), \alpha \rangle$ *is defined then* $A(\alpha)(u)$ *is defined,*

R *is a total function from* $\mathcal{R}$ *to the powerset of* $\overline{U}^*$, *and*

> *for each* $\rho \in \mathcal{R}$, $R(\rho) \subseteq \overline{U}^{\mathcal{AR}(\rho)}$.

4 Definition. $\langle \mathsf{Chain}, \sqsubseteq^c \rangle$ *is the smallest partial order such that*

$\mathsf{Chain} = \{chain, echain, nechain\}$,

echain $\sqsubseteq^c$ *chain*, *nechain* $\sqsubseteq^c$ *chain*.

5 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$,

$\widehat{\mathcal{G}} = \mathcal{G} \cup \mathsf{Chain} \cup \{metatop\}$,

$\widehat{\sqsubseteq} = \sqsubseteq \cup \sqsubseteq^c \cup \left\{ \langle \sigma, metatop \rangle \,\middle|\, \sigma \in \widehat{\mathcal{G}} \right\}$,[1]

$\widehat{\mathcal{S}} = \mathcal{S} \cup \{echain, nechain\}$, *and*

$\widehat{\mathcal{A}} = \mathcal{A} \cup \{\dagger, \triangleright\}$.

6 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$,

$\widehat{S}$ *is the total function from* $\overline{U}$ *to* $\widehat{\mathcal{S}}$ *such that*

for each $u \in U$, $\widehat{S}(u) = S(u)$,
for each $u_1 \in U$, ..., *for each* $u_n \in U$,

$$\widehat{S}(\langle u_1, \ldots, u_n \rangle) = \begin{cases} echain & \text{if } n = 0, \\ nechain & \text{if } n > 0 \end{cases}, \text{ and}$$

$\widehat{A}$ *is the partial function from* $\widehat{\mathcal{A}}$ *to the set of partial functions from* $\overline{U}$ *to* $\overline{U}$ *such that*

for each $\alpha \in \mathcal{A}$, $\widehat{A}(\alpha) = A(\alpha)$, *and*

$\widehat{A}(\dagger)$ *is the total function from* U^+ *to* U *such that for each* $\langle u_0, \ldots, u_n \rangle \in U^+$,

$\widehat{A}(\dagger)(\langle u_0, \ldots, u_n \rangle) = u_0$, *and*

$\widehat{A}(\triangleright)$ *is the total function from* U^+ *to* U^* *such that for each* $\langle u_0, \ldots, u_n \rangle \in U^+$,

$\widehat{A}(\triangleright)(\langle u_0, \ldots, u_n \rangle) = \langle u_1, \ldots, u_n \rangle$.

7 Definition. *For each signature* Σ, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$,

$Ass_I = \overline{U}^{\mathsf{Var}}$ *is the set of variable assignments in* I.

8 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, $\mathcal{T}^\Sigma$ *is the smallest set such that*

$: \in \mathcal{T}^\Sigma$,

for each $v \in \mathsf{Var}$, $v \in \mathcal{T}^\Sigma$, *and*

for each $\alpha \in \widehat{\mathcal{A}}$ *and each* $\tau \in \mathcal{T}^\Sigma$, $\tau\alpha \in \mathcal{T}^\Sigma$.

[1] Note that $\left\langle \widehat{\mathcal{G}}, \widehat{\sqsubseteq} \right\rangle$ is a finite partial order.

9 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $\mathrm{ass} \in Ass_I$, T_I^{ass} *is the total function from* $\mathcal{T}^\Sigma$ *to the set of partial functions from* U *to* $\overline{U}$ *such that for each* $u \in U$,

$T_I^{\mathrm{ass}}(:)(u)$ *is defined and* $T_I^{\mathrm{ass}}(:)(u) = u$,

for each $v \in \mathsf{Var}$, $T_I^{\mathrm{ass}}(v)(u)$ *is defined and* $T_I^{\mathrm{ass}}(v)(u) = \mathrm{ass}(v)$,

for each $\tau \in \mathcal{T}^\Sigma$, *for each* $\alpha \in \widehat{\mathcal{A}}$,

> $T_I^{\mathrm{ass}}(\tau\alpha)(u)$ *is defined*
> *iff* $T_I^{\mathrm{ass}}(\tau)(u)$ *is defined and* $\widehat{A}(\alpha)(T_I^{\mathrm{ass}}(\tau)(u))$ *is defined, and*
>
> *if* $T_I^{\mathrm{ass}}(\tau\alpha)(u)$ *is defined*
>
> *then* $T_I^{\mathrm{ass}}(\tau\alpha)(u) = \widehat{A}(\alpha)(T_I^{\mathrm{ass}}(\tau)(u))$.

10 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, $\mathcal{D}^\Sigma$ *is the smallest set such that*

for each $\sigma \in \widehat{\mathcal{G}}$, *for each* $\tau \in \mathcal{T}^\Sigma$, $\tau{\sim}\sigma \in \mathcal{D}^\Sigma$,

for each $\tau_1 \in \mathcal{T}^\Sigma$, *for each* $\tau_2 \in \mathcal{T}^\Sigma$, $\tau_1{\approx}\tau_2 \in \mathcal{D}^\Sigma$,

for each $\rho \in \mathcal{R}$, *for each* $x_1 \in \mathsf{Var}$, ..., *for each* $x_{\mathcal{AR}(\rho)} \in \mathsf{Var}$, $\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)}) \in \mathcal{D}^\Sigma$,

for each $x \in \mathsf{Var}$, *for each* $\delta \in \mathcal{D}^\Sigma$, $\exists x\, \delta \in \mathcal{D}^\Sigma$,

for each $x \in \mathsf{Var}$, *for each* $\delta \in \mathcal{D}^\Sigma$, $\forall x\, \delta \in \mathcal{D}^\Sigma$,

for each $\delta \in \mathcal{D}^\Sigma$, $\neg\delta \in \mathcal{D}^\Sigma$,

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $[\delta_1 \wedge \delta_2] \in \mathcal{D}^\Sigma$,

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $[\delta_1 \vee \delta_2] \in \mathcal{D}^\Sigma$,

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $[\delta_1 \rightarrow \delta_2] \in \mathcal{D}^\Sigma$, *and*

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $[\delta_1 \leftrightarrow \delta_2] \in \mathcal{D}^\Sigma$.

11 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$,

$FV(:) = \emptyset$,

for each $v \in \mathsf{Var}$, $FV(v) = \{v\}$,

for each $\tau \in \mathcal{T}^\Sigma$, *for each* $\alpha \in \widehat{\mathcal{A}}$, $FV(\tau\alpha) = FV(\tau)$,

for each $\tau \in \mathcal{T}^\Sigma$, *for each* $\sigma \in \widehat{\mathcal{G}}$, $FV(\tau \sim \sigma) = FV(\tau)$,

for each $\tau_1, \tau_2 \in \mathcal{T}^\Sigma$, $FV(\tau_1 \approx \tau_2) = FV(\tau_1) \cup FV(\tau_2)$,

for each $\rho \in \mathcal{R}$, *for each* $x_1, \ldots, x_{\mathcal{AR}(\rho)} \in \mathsf{Var}$, $FV(\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)})) = \{x_1, \ldots, x_{\mathcal{AR}(\rho)}\}$,

for each $\delta \in \mathcal{D}^\Sigma$, *for each* $v \in \mathsf{Var}$, $FV(\exists v\, \delta) = FV(\delta) \backslash \{v\}$,

for each $\delta \in \mathcal{D}^\Sigma$, *for each* $v \in \mathsf{Var}$, $FV(\forall v\, \delta) = FV(\delta) \backslash \{v\}$,

for each $\delta \in \mathcal{D}^\Sigma$, $FV(\neg\delta) = FV(\delta)$,
for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \wedge \delta_2) = FV(\delta_1) \cup FV(\delta_2)$,
for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \vee \delta_2) = FV(\delta_1) \cup FV(\delta_2)$,
for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \rightarrow \delta_2) = FV(\delta_1) \cup FV(\delta_2)$, *and*
for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$, $FV(\delta_1 \leftrightarrow \delta_2) = FV(\delta_1) \cup FV(\delta_2)$.

12 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *and for each* $u \in U$,

$$Co^u_I = \left\{ u' \in U \;\middle|\; \begin{array}{l} \textit{for some } \mathrm{ass} \in Ass_I, \\ \textit{for some } \pi \in \mathcal{A}^*, \\ \quad T^{\mathrm{ass}}_I(:\pi)(u) \textit{ is defined, and} \\ \quad u' = T^{\mathrm{ass}}_I(:\pi)(u) \end{array} \right\}.$$

We call Co^u_I the set of components of u in I.

13 Definition. *For each signature* Σ, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $\mathrm{ass} \in Ass_I$, *for each* $v \in \mathsf{Var}$, *for each* $w \in \mathsf{Var}$, *for each* $u \in \overline{U}$,

$$\mathrm{ass}\tfrac{u}{v}(w) = \begin{cases} u & \textit{if } v = w \\ \mathrm{ass}(w) & \textit{otherwise.} \end{cases}$$

14 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $\mathrm{ass} \in Ass_I$, D^{ass}_I *is the total function from* $\mathcal{D}^\Sigma$ *to the powerset of* U *such that*

for each $\tau \in \mathcal{T}^\Sigma$, *for each* $\sigma \in \widehat{\mathcal{G}}$,

$$D^{\mathrm{ass}}_I(\tau \sim \sigma) = \left\{ u \in U \;\middle|\; \begin{array}{l} T^{\mathrm{ass}}_I(\tau)(u) \textit{ is defined, and} \\ \widehat{S}(T^{\mathrm{ass}}_I(\tau)(u)) \mathrel{\widehat{\sqsubseteq}} \sigma \end{array} \right\},$$

for each $\tau_1 \in \mathcal{T}^\Sigma$, *for each* $\tau_2 \in \mathcal{T}^\Sigma$,

$$D^{\mathrm{ass}}_I(\tau_1 \approx \tau_2) = \left\{ u \in U \;\middle|\; \begin{array}{l} T^{\mathrm{ass}}_I(\tau_1)(u) \textit{ is defined,} \\ T^{\mathrm{ass}}_I(\tau_2)(u) \textit{ is defined, and} \\ T^{\mathrm{ass}}_I(\tau_1)(u) = T^{\mathrm{ass}}_I(\tau_2)(u) \end{array} \right\},$$

for each $\rho \in \mathcal{R}$, *for each* $x_1 \in \mathsf{Var}$, ..., *for each* $x_{\mathcal{AR}(\rho)} \in \mathsf{Var}$,

$$\begin{array}{l} D^{\mathrm{ass}}_I(\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)})) \\ = \left\{ u \in U \;\middle|\; \left\langle \mathrm{ass}(x_1), \ldots, \mathrm{ass}(x_{\mathcal{AR}(\rho)}) \right\rangle \in R(\rho) \right\}, \end{array}$$

for each $v \in \mathsf{Var}$, *for each* $\delta \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}(\exists v\ \delta) = \left\{ u \in U \;\middle|\; \begin{array}{l} \textit{for some } u' \in \overline{Co_I^u}, \\ u \in D_I^{\text{ass}\frac{u'}{v}}(\delta) \end{array} \right\},$$

for each $v \in \mathsf{Var}$, *for each* $\delta \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}(\forall v\ \delta) = \left\{ u \in U \;\middle|\; \begin{array}{l} \textit{for each } u' \in \overline{Co_I^u}, \\ u \in D_I^{\text{ass}\frac{u'}{v}}(\delta) \end{array} \right\},$$

for each $\delta \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}(\neg\delta) = U \backslash D_I^{\text{ass}}(\delta),$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}([\delta_1 \wedge \delta_2]) = D_I^{\text{ass}}(\delta_1) \cap D_I^{\text{ass}}(\delta_2),$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}([\delta_1 \vee \delta_2]) = D_I^{\text{ass}}(\delta_1) \cup D_I^{\text{ass}}(\delta_2),$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}([\delta_1 \rightarrow \delta_2]) = (U \backslash D_I^{\text{ass}}(\delta)) \cup D_I^{\text{ass}}(\delta_2), \textit{ and}$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$D_I^{\text{ass}}([\delta_1 \leftrightarrow \delta_2]) = ((U \backslash D_I^{\text{ass}}(\delta_1)) \cap (U \backslash D_I^{\text{ass}}(\delta_2))) \cup (D_I^{\text{ass}}(\delta_1) \cap D_I^{\text{ass}}(\delta_2)).$$

15 Proposition. *For each signature* Σ, *for each* Σ *interpretation* I, *for each* $\text{ass}_1 \in Ass_I$, *for each* $\text{ass}_2 \in Ass_I$,

for each $\tau \in \mathcal{T}^\Sigma$,

if for each $v \in FV(\tau)$, $\text{ass}_1(v) = \text{ass}_2(v)$ *then* $T_I^{\text{ass}_1}(\tau) = T_I^{\text{ass}_2}(\tau)$, *and*

for each $\delta \in \mathcal{D}^\Sigma$,

if for each $v \in FV(\delta)$, $\text{ass}_1(v) = \text{ass}_2(v)$ *then* $D_I^{\text{ass}_1}(\delta) = D_I^{\text{ass}_2}(\delta)$.

16 Definition. *For each signature* Σ,

$$\mathcal{D}_0^\Sigma = \{\delta \in \mathcal{D}^\Sigma \mid FV(\delta) = \emptyset\}.$$

17 Corollary. *For each signature* Σ, *for each* $\delta \in \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* I, *for each* $\text{ass}_1 \in Ass_I$, *for each* $\text{ass}_2 \in Ass_2$,

$D_I^{\mathrm{ass}_1}(\delta) = D_I^{\mathrm{ass}_2}(\delta)$.

18 Definition. *For each signature* Σ*, for each* Σ *interpretation* $I = \langle U, S, A, R\rangle$*,* D_I *is the total function from* $\mathcal{D}_0^\Sigma$ *to the powerset of* U *such that for each* $\delta \in \mathcal{D}_0^\Sigma$*,*

$$D_I(\delta) = \left\{ u \in U \,\middle|\, \begin{array}{l} \textit{for each } \mathrm{ass} \in Ass_I, \\ u \in D_I^{\mathrm{ass}}(\delta) \end{array} \right\}.$$

19 Definition. *For each signature* Σ*, for each* Σ *interpretation* $I = \langle U, S, A, R\rangle$*,* Θ_I *is the total function from the powerset of* $\mathcal{D}_0^\Sigma$ *to the powerset of* U *such that for each* $\theta \subseteq \mathcal{D}_0^\Sigma$*,*

$$\Theta_I(\theta) = \left\{ u \in U \,\middle|\, \begin{array}{l} \textit{for each } \delta \in \theta, \\ u \in D_I(\delta) \end{array} \right\}.$$

20 Definition. Γ *is a grammar iff*

Γ *is a pair* $\langle \Sigma, \theta \rangle$*,*

Σ *is a signature, and*

$\theta \subseteq \mathcal{D}_0^\Sigma$.

21 Definition. *For each grammar* $\Gamma = \langle \Sigma, \theta\rangle$*, for each* Σ *interpretation* $I = \langle U, S, A, R\rangle$*,*

I *is a* Γ *model iff* $\Theta_I(\theta) = U$.

22 Definition. *For each grammar* $\Gamma = \langle \Sigma, \theta\rangle$*, for each* Σ *interpretation* I*,*

I *is an exhaustive* Γ *model iff*

> I *is a* Γ *model, and*
> *for each* $\theta' \subseteq \mathcal{D}_0^\Sigma$*, for each* Σ *interpretation* I'*,*
>
> > *if* I' *is a* Γ *model and* $\Theta_{I'}(\theta') \neq \emptyset$*,*
> > *then* $\Theta_I(\theta') \neq \emptyset$.

2 A Different Characterization of Exhaustive Models

Suppose that S is a set. We write $\overline{S}^*$ for $(\overline{S})^*$ and S^{**} for $(S^*)^*$.

23 Definition. *For each signature Σ, for each Σ interpretation $I_1 = \langle U_1, S_1, A_1, R_1\rangle$, for each $u_1 \in U_1$, for each Σ interpretation $I_2 = \langle U_2, S_2, A_2, R_2\rangle$, for each $u_2 \in U_2$,*

f is a congruence from $\langle u_1, I_1\rangle$ to $\langle u_2, I_2\rangle$ in Σ
iff f is a bijection from $\overline{Co_{I_1}^{u_1}}$ to $\overline{Co_{I_2}^{u_2}}$,
for each $u \in \overline{Co_{I_1}^{u_1}}$, $\widehat{S_1}(u) = \widehat{S_2}(f(u))$,
for each $\alpha \in \widehat{\mathcal{A}}$, for each $u \in \overline{Co_{I_1}^{u_1}}$,
$\widehat{A_1}(\alpha)(u)$ is defined iff $\widehat{A_2}(\alpha)(f(u))$ is defined, and
if $\widehat{A_1}(\alpha)(u)$ is defined then $f(\widehat{A_1}(\alpha)(u)) = \widehat{A_2}(\alpha)(f(u))$,
for each $\rho \in \mathcal{R}$, for each $o_1 \in \overline{Co_{I_1}^{u_1}}, \ldots,$ for each $o_{\mathcal{AR}(\rho)} \in \overline{Co_{I_1}^{u_1}}$,
$\langle o_1, \ldots, o_{\mathcal{AR}(\rho)}\rangle \in R_1(\rho)$ iff $\langle f(o_1), \ldots, f(o_{\mathcal{AR}(\rho)})\rangle \in R_2(\rho)$, and
$f(u_1) = u_2$.

A first object in a first interpretation and a second object in a second interpretation are congruent iff there is a species, attribute and relation preserving bijection from the components of the first object in the first interpretation to the components of the second object in the second interpretation such that the bijection maps the first object to the second object.

24 Definition. *For each signature Σ, for each Σ interpretation $I_1 = \langle U_1, S_1, A_1, R_1\rangle$, for each $u_1 \in U_1$, for each Σ interpretation $I_2 = \langle U_2, S_2, A_2, R_2\rangle$, for each $u_2 \in U_2$,*

$\langle u_1, I_1\rangle$ and $\langle u_2, I_2\rangle$ are congruent in Σ
iff for some f, f is a congruence from $\langle u_1, I_1\rangle$ to $\langle u_2, I_2\rangle$ in Σ.

25 Definition. *For each signature Σ, for each Σ interpretation $I_1 = \langle U_1, S_1, A_1, R_1\rangle$, for each $u_1 \in U_1$, for each Σ interpretation $I_2 = \langle U_2, S_2, A_2, R_2\rangle$, for each $u_2 \in U_2$,*

$\langle u_1, I_1\rangle$ and $\langle u_2, I_2\rangle$ are indiscernible in Σ
iff for each $\delta \in \mathcal{D}_0^\Sigma$, $u_1 \in D_{I_1}(\delta)$ iff $u_2 \in D_{I_2}(\delta)$.

We use a standard definition of functional composition:

26 Definition. *For each set U, for each set V, for each set W, for each total function f from U to V, for each total function g from V to W,*

$g \circ f$ is the function from U to W such that, for each $u \in U$,

$g \circ f(u) = g(f(u))$.

By proposition 15 the following definition is well-made.

27 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$*, for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$*,* T_I *is the partial function from the Cartesian product of* $\overline{\mathcal{A}^*}$ *and* U *to* $\overline{U}$ *such that*

for each $\pi \in \mathcal{A}^*$*, for each* $u \in U$*,*

$T_I(\pi, u)$ *is defined iff for some* $\mathrm{ass} \in Ass_I$*,* $T_I^{\mathrm{ass}}(:\pi)(u)$ *is defined, and*
if $T_I(\pi, u)$ *is defined then for some* $\mathrm{ass} \in Ass_I$*,* $T_I(\pi, u) = T_I^{\mathrm{ass}}(:\pi)(u)$*,*

for each $\langle \pi_1, \ldots, \pi_n \rangle \in \mathcal{A}^{**}$*, for each* $u \in U$*,*

$T_I(\langle \pi_1, \ldots, \pi_n \rangle, u)$ *is defined*
iff $T_I(\pi_1, u)$ *is defined, ...,* $T_I(\pi_n, u)$ *is defined, and*
if $T_I(\langle \pi_1, \ldots, \pi_n \rangle, u)$ *is defined*
then $T_I(\langle \pi_1, \ldots, \pi_n \rangle, u) = \langle T_I(\pi_1, u), \ldots, T_I(\pi_n, u) \rangle$*.*

28 Proposition. *For each signature* Σ*, for each* Σ *interpretation* $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$*, for each* $o_1 \in U_1$*, for each* Σ *interpretation* $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$*, for each* $o_2 \in U_2$*,*

$\langle o_1, I_1 \rangle$ *and* $\langle o_2, I_2 \rangle$ *are congruent in* Σ *iff* $\langle o_1, I_1 \rangle$ *and* $\langle o_2, I_2 \rangle$ *are indiscernible in* Σ*.*

Proof. *Firstly, for each signature* Σ*, for each* Σ *interpretation* $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$*, for each* $o_1 \in U_1$*, for each* Σ *interpretation* $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$*, for each* $o_2 \in U_2$*, for each congruence* f *from* $\langle o_1, I_1 \rangle$ *to* $\langle o_2, I_2 \rangle$*,*

for each $\tau \in \mathcal{T}^\Sigma$*, for each total function* ass *from* Var *to* $\overline{Co_{I_1}^{o_1}}$*,*

$T_{I_1}^{\mathrm{ass}}(\tau)(o_1)$ *is defined iff* $T_{I_2}^{f \circ \mathrm{ass}}(\tau)(o_2)$ *is defined, and*
if $T_{I_1}^{\mathrm{ass}}(\tau)(o_1)$ *is defined then* $f(T_{I_1}^{\mathrm{ass}}(\tau)(o_1)) = T_{I_2}^{f \circ \mathrm{ass}}(\tau)(o_2)$*, and*
by induction on the length of τ

for each $\delta \in \mathcal{D}^\Sigma$*, for each total function* ass *from* Var *to* $\overline{Co_{I_1}^{o_1}}$*,*

$o_1 \in D_{I_1}^{\mathrm{ass}}(\delta)$ *iff* $o_2 \in D_{I_2}^{f \circ \mathrm{ass}}(\delta)$*.* *by induction on the complexity of* δ
(since, for each $o \in U$*,* $f \circ (\mathrm{ass}\frac{o}{x}) = (f \circ \mathrm{ass})\frac{f(o)}{x}$*)*

Thus, for each signature Σ*, for each* Σ *interpretation* $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$*, for each* $o_1 \in U_1$*, for each* Σ *interpretation* $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$*, for each* $o_2 \in U_2$*,*

$\langle o_1, I_1 \rangle$ *and* $\langle o_2, I_2 \rangle$ *are congruent in* Σ

$\Longrightarrow$ *for some congruence* f*,* f *is a congruence from* $\langle o_1, I_1 \rangle$ *to* $\langle o_2, I_2 \rangle$ *in* Σ

$\Longrightarrow$ *for each* $\delta \in \mathcal{D}_0^\Sigma$*,*

$o_1 \in D_{I_1}(\delta)$

$\iff$ *for each* ass $\in Ass_{I_1}$, $o_1 \in D^{\text{ass}}_{I_1}(\delta)$

$\iff$ *for some* ass $\in Ass_{I_1}$,

$o_1 \in D^{\text{ass}}_{I_1}(\delta)$ *and, for each* $v \in \mathsf{Var}$, $\text{ass}(v) \in \overline{Co^{o_1}_{I_1}}$ *by corollary 17*

$\iff$ *for some* ass $\in Ass_{I_2}$,

$o_2 \in D^{\text{ass}}_{I_2}(\delta)$ *and, for each* $v \in \mathsf{Var}$, $\text{ass}(v) \in \overline{Co^{o_2}_{I_2}}$

$\iff$ *for each* ass $\in Ass_{I_2}$, $o_2 \in D^{\text{ass}}_{I_2}(\delta)$

$\iff o_2 \in D_{I_2}(\delta)$

$\Longrightarrow \langle o_1, I_1\rangle$ *and* $\langle o_2, I_2\rangle$ *are indiscernible in* Σ.

Secondly, suppose $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR}\rangle$ *is a signature,* $I_1 = \langle U_1, S_1, A_1, R_1\rangle$ *is a* Σ *interpretation,* $o_1 \in U_1$, $I_2 = \langle U_2, S_2, A_2, R_2\rangle$ *is a* Σ *interpretation,* $o_2 \in U_2$, *and* $\langle o_1, I_1\rangle$ *and* $\langle o_2, I_2\rangle$ *are indiscernible in* Σ.

Firstly, for each $\pi \in \mathcal{A}^*$, $T_{I_1}(\pi, o_1)$ *is defined iff* $T_{I_2}(\pi, o_2)$ *is defined.*
by induction on the length of π

Secondly, let $f = \left\{ \langle o'_1, o'_2\rangle \in Co^{o_1}_{I_1} \times Co^{o_2}_{I_2} \;\middle|\; \begin{array}{l} \text{for some } \pi \in \mathcal{A}^* \\ \quad T_{I_1}(\pi, o_1) \text{ is defined,} \\ \quad T_{I_2}(\pi, o_2) \text{ is defined,} \\ \quad o'_1 = T_{I_1}(\pi, o_1), \text{ and} \\ \quad o'_2 = T_{I_2}(\pi, o_2) \end{array} \right\}$.

f *is a bijection from* $Co^{o_1}_{I_1}$ *to* $Co^{o_2}_{I_2}$.

Let $\overline{f}$ *be the total function from* $\overline{Co^{o_1}_{I_1}}$ *to* $\overline{Co^{o_2}_{I_2}}$ *such that,*

for each $u \in Co^{o_1}_{I_1}$, $\overline{f}(u) = f(u)$, *and*
for each $\langle u_1, \ldots, u_n\rangle \in (Co^{o_1}_{I_1})^*$, $\overline{f}(\langle u_1, \ldots, u_n\rangle) = \langle f(u_1), \ldots, f(u_n)\rangle$.

Clearly, $\overline{f}$ *is a bijection.*

Thirdly, for each $u \in Co^{o_1}_{I_1}$, $S_1(u) = S_2(f(u))$. *Thus, for each* $u \in \overline{Co^{o_1}_{I_1}}$, $\widehat{S_1}(u) = \widehat{S_2}(\overline{f}(u))$.

Fourthly, for each $\alpha \in \mathcal{A}$, *for each* $u \in Co^{o_1}_{I_1}$,

$A_1(\alpha)(u)$ *is defined iff* $A_2(\alpha)(f(u))$ *is defined, and*
if $A_1(\alpha)(u)$ *is defined then* $f(A_1(\alpha)(u)) = A_2(\alpha)(f(u))$.

Thus, for each $\alpha \in \widehat{\mathcal{A}}$, *for each* $u \in \overline{Co^{o_1}_{I_1}}$,

$\widehat{A_1}(\alpha)(u)$ *is defined iff* $\widehat{A_2}(\alpha)(\overline{f}(u))$ *is defined, and*
if $\widehat{A_1}(\alpha)(u)$ *is defined then* $\overline{f}(\widehat{A_1}(\alpha)(u)) = \widehat{A_2}(\alpha)(\overline{f}(u))$.

Fifthly, for each $\rho \in \mathcal{R}$*, for each* $u_1 \in \overline{Co^{o_1}_{I_1}}$*, ..., for each* $u_n \in \overline{Co^{o_1}_{I_1}}$*,*

$$\begin{aligned}
&\langle u_1, \ldots, u_n \rangle \in R_{I_1}(\rho),\\
\iff &\text{for some } \pi_1 \in \overline{\mathcal{A}^*}, \ldots, \text{for some } \pi_n \in \overline{\mathcal{A}^*},\\
&u_1 = T_{I_1}(\pi_1, o_1), \ldots, u_n = T_{I_1}(\pi_n, o_1), \text{ and } \langle u_1, \ldots, u_n \rangle \in R_{I_1}(\rho)\\
\iff &\text{for some } \pi_1 \in \overline{\mathcal{A}^*}, \ldots, \text{for some } \pi_n \in \overline{\mathcal{A}^*},\\
&u_1 = T_{I_1}(\pi_1, o_1), \ldots, u_n = T_{I_1}(\pi_n, o_1), \text{ and}\\
&o_1 \in D_{I_1}(\exists x_1 \ldots \exists x_n \ (\rho(x_1, \ldots, x_n) \ \wedge \ x_1 \approx :\pi_1 \ \wedge \ldots \wedge x_n \approx :\pi_n))^2\\
\iff &\text{for some } \pi_1 \in \overline{\mathcal{A}^*}, \ldots, \text{for some } \pi_n \in \overline{\mathcal{A}^*},\\
&\overline{f}(u_1) = T_{I_2}(\pi_1, o_2), \ldots, \overline{f}(u_n) = T_{I_2}(\pi_n, o_2), \text{ and}\\
&o_2 \in D_{I_2}(\exists x_1 \ldots \exists x_n \ (\rho(x_1, \ldots, x_n) \ \wedge \ x_1 \approx :\pi_1 \ \wedge \ldots \wedge x_n \approx :\pi_n))\\
\iff &\text{for some } \pi_1 \in \overline{\mathcal{A}^*}, \ldots, \text{for some } \pi_n \in \overline{\mathcal{A}^*},\\
&\overline{f}(u_1) = T_{I_2}(\pi_1, o_2), \ldots, \overline{f}(u_n) = T_{I_2}(\pi_n, o_2), \text{ and } \langle \overline{f}(u_1), \ldots, \overline{f}(u_n) \rangle \in R_{I_2}(\rho)\\
\iff &\langle \overline{f}(u_1), \ldots, \overline{f}(u_n) \rangle \in R_{I_2}(\rho).
\end{aligned}$$

Finally, $\overline{f}(o_1) = o_2$*.*

Therefore, $\overline{f}$ *is a congruence from* $\langle o_1, I_1 \rangle$ *to* $\langle o_2, I_2 \rangle$ *in* Σ*.* $\langle o_1, I_1 \rangle$ *and* $\langle o_2, I_2 \rangle$ *are, thus, congruent in* Σ*.* ■

29 Definition. *For each signature* Σ*, for each* Σ *interpretation* $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$*, for each* Σ *interpretation* $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$*,*

I_1 *simulates* I_2 *in* Σ

iff for each $u_2 \in U_2$*, for some* $u_1 \in U_1$*,* $\langle u_1, I_1 \rangle$ *and* $\langle u_2, I_2 \rangle$ *are congruent in* Σ*.*

A Σ interpretation I_1 simulates Σ interpretation I_2 just in case every object in I_2 has a congruent counterpart in I_1.

30 Proposition. *For each signature* Σ*,*

for each Σ *interpretation* I*,*

I *simulates* I *in* Σ*, and*

for each Σ *interpretation* I_1*, for each* Σ *interpretation* I_2*, for each* Σ *interpretation* I_3*,*

if I_1 *simulates* I_2 *in* Σ *and* I_2 *simulates* I_3 *in* Σ *then* I_1 *simulates* I_3 *in* Σ*.*

31 Proposition. *For each signature* Σ*, for each* $\theta \subseteq \mathcal{D}^{\Sigma}_0$*, for each* Σ *interpretation* I*,*

[2] We write $x \approx :\langle \pi_1, \ldots, \pi_n \rangle$ as an abbreviation for $x\dagger \approx :\pi_1 \ \wedge \ldots \wedge \ x \triangleright^i \dagger \approx :\pi_{i+1} \ \wedge \ldots \wedge \ x \triangleright^n \sim \textit{echain}$

for each Σ *interpretation* I',

if I' *is a* $\langle \Sigma, \theta \rangle$ *model then* I *simulates* I' *in* Σ

iff for each $\theta' \subseteq \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* I',

if I' *is a* $\langle \Sigma, \theta \rangle$ *model and* $\Theta_{I'}(\theta') \neq \emptyset$ *then* $\Theta_I(\theta') \neq \emptyset$.

Proof. *Firstly, for each signature* Σ, *for each* $\theta \subseteq \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$,

for each Σ *interpretation* $I' = \langle U', S', A', R' \rangle$,
if I' *is a* $\langle \Sigma, \theta \rangle$ *model then* I *simulates* I'

$\Longrightarrow$ *for each* $\theta' \subseteq \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* $I' = \langle U', S', A', R' \rangle$,
I' *is a* $\langle \Sigma, \theta \rangle$ *model and* $\Theta_{I'}(\theta') \neq \emptyset$

$\Longrightarrow$ I *simulates* I' *in* Σ *and, for some* $u' \in U'$, $u' \in \Theta_{I'}(\theta')$

$\Longrightarrow$ *for some* $u \in U$, *for some* $u' \in U'$,
$\langle u, I \rangle$ *and* $\langle u', I' \rangle$ *are congruent in* Σ *and* $u' \in \Theta_{I'}(\theta')$

$\Longrightarrow$ *for some* $u \in U$, *for some* $u' \in U'$,
$\langle u, I \rangle$ *and* $\langle u', I' \rangle$ *are indiscernible in* Σ *and* $u' \in \Theta_{I'}(\theta')$ *by proposition 28*

$\Longrightarrow$ *for some* $u \in U$, $u \in \Theta_I(\theta')$

$\Longrightarrow$ $\Theta_I(\theta') \neq \emptyset$.

Secondly, for each signature Σ, *for each* $\theta \subseteq \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $\theta' \subseteq \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* I',
if I' *is a* $\langle \Sigma, \theta \rangle$ *model and* $\Theta_{I'}(\theta') \neq \emptyset$ *then* $\Theta_I(\theta') \neq \emptyset$

$\Longrightarrow$ *for each* Σ *interpretation* $I' = \langle U', S', A', R' \rangle$,
I' *is a* $\langle \Sigma, \theta \rangle$ *model*

$\Longrightarrow$ *for each* u',

$u' \in U'$

$\Longrightarrow u' \in \Theta_{I'} \left\{ \delta \in \mathcal{D}_0^\Sigma \,\middle|\, u' \in D_{I'}(\delta) \right\}$

$\Longrightarrow \Theta_{I'} \left\{ \delta \in \mathcal{D}_0^\Sigma \,\middle|\, u' \in D_{I'}(\delta) \right\} \neq \emptyset$

$\Longrightarrow \Theta_{I} \left\{ \delta \in \mathcal{D}_0^\Sigma \,\middle|\, u' \in D_{I'}(\delta) \right\} \neq \emptyset$

$\Longrightarrow$ *for some* $u \in U$, $u \in \Theta_{I} \left\{ \delta \in \mathcal{D}_0^\Sigma \,\middle|\, u' \in D_{I'}(\delta) \right\}$

$\Longrightarrow$ *for some* $u \in U$, *for each* $\delta \in \mathcal{D}_0^\Sigma$,

$u \in D_I(\delta)$

$\Longrightarrow u \notin D_I(\neg\delta)$

$\Longrightarrow \neg\delta \notin \left\{ \delta \in \mathcal{D}_0^\Sigma \,\middle|\, u' \in D_{I'}(\delta) \right\}$

$\Longrightarrow u' \notin D_{I'}(\neg\delta)$

$\Longrightarrow u' \in D_{I'}(\delta)$, *and*

$u' \in D_{I'}(\delta)$

$\Longrightarrow \delta \in \left\{ \delta \in \mathcal{D}_0^\Sigma \,\middle|\, u' \in D_{I'}(\delta) \right\}$

$\Longrightarrow u \in D_I(\delta)$

$\Longrightarrow$ *for some* $u \in U$, $\langle u, I\rangle$ *and* $\langle u', I'\rangle$ *are indiscernible in* Σ

$\Longrightarrow$ *for some* $u \in U$, $\langle u, I\rangle$ *and* $\langle u', I'\rangle$ *are congruent in* Σ *by proposition 28*

$\Longrightarrow I$ *simulates* I' *in* Σ.

■

32 Theorem. *For each signature* Σ, *for each* $\theta \subseteq \mathcal{D}_0^\Sigma$, *for each* Σ *interpretation* I,

I is an exhaustive $\langle\Sigma, \theta\rangle$ model
iff I is a $\langle\Sigma, \theta\rangle$ model, and for each Σ interpretation I',

if I' is a $\langle\Sigma, \theta\rangle$ model then I simulates I' in Σ.

3 Existence Proof for Exhaustive Models

In this section we show that, for each grammar, there exists an exhaustive model. For each grammar $\langle\Sigma, \theta\rangle$, we construct a Σ interpretation which we call the canonical Σ interpretation of θ. We then show that the canonical Σ interpretation is an exhaustive model of the grammar $\langle\Sigma, \theta\rangle$.

Suppose $\Sigma = \langle\mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR}\rangle$ is a signature. We call each member of $\mathcal{A}^*$ a Σ path, and write ε for the empty path, the unique path of length zero.

33 Definition. *For each signature* $\Sigma = \langle\mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR}\rangle$,

μ is a morph in Σ

iff μ is a quadruple $\langle\beta, \varrho, \lambda, \xi\rangle$,

$\beta \subseteq \mathcal{A}^*$,

$\varepsilon \in \beta$,

for each $\pi \in \mathcal{A}^*$, *for each* $\alpha \in \mathcal{A}$,

if $\pi\alpha \in \beta$ *then* $\pi \in \beta$,

ϱ *is an equivalence relation over* β,

for each $\pi_1 \in \mathcal{A}^*$, *for each* $\pi_2 \in \mathcal{A}^*$, *for each* $\alpha \in \mathcal{A}$, *if* $\pi_1\alpha \in \beta$ *and* $\langle \pi_1, \pi_2 \rangle \in \varrho$ *then* $\langle \pi_1\alpha, \pi_2\alpha \rangle \in \varrho$,

λ *is a total function from* β *to* $\mathcal{S}$,

for each $\pi_1 \in \mathcal{A}^*$, *for each* $\pi_2 \in \mathcal{A}^*$,

if $\langle \pi_1, \pi_2 \rangle \in \varrho$ *then* $\lambda(\pi_1) = \lambda(\pi_2)$,

for each $\pi \in \mathcal{A}^*$, *for each* $\alpha \in \mathcal{A}$,

if $\pi\alpha \in \beta$ *then* $\mathcal{F}\langle \lambda(\pi), \alpha \rangle$ *is defined and* $\lambda(\pi\alpha) \sqsubseteq \mathcal{F}\langle \lambda(\pi), \alpha \rangle$,

for each $\pi \in \mathcal{A}^*$, *for each* $\alpha \in \mathcal{A}$,

if $\pi \in \beta$ *and* $\mathcal{F}\langle \lambda(\pi), \alpha \rangle$ *is defined then* $\pi\alpha \in \beta$,

$\xi \subseteq \mathcal{R} \times \overline{\beta}^*$,

for each $\rho \in \mathcal{R}$, *for each* $\pi_1 \in \overline{\beta}$,..., *for each* $\pi_n \in \overline{\beta}$,

if $\langle \rho, \pi_1, \ldots, \pi_n \rangle \in \xi$ *then* $n = \mathcal{AR}(\rho)$, *and*

for each $\rho \in \mathcal{R}$, *for each* $\pi_1 \in \overline{\beta}$, ..., *for each* $\pi_n \in \overline{\beta}$, *for each* $\pi'_1 \in \overline{\beta}$, ..., *for each* $\pi'_n \in \overline{\beta}$,

if $\langle \rho, \pi_1, \ldots, \pi_n \rangle \in \xi$, *and*
for each $i \in \{1, \ldots, n\}$,
$\pi_i \in \beta$ *and* $\langle \pi_i, \pi'_i \rangle \in \varrho$, *or*
for some $m \in \mathbb{N}$,
$\pi_i \in \beta^*$,
$\pi_i = \langle \pi_{i_1}, \ldots, \pi_{i_m} \rangle$,
$\pi'_i = \langle \pi'_{i_1}, \ldots, \pi'_{i_m} \rangle$, *and*
$\langle \pi_{i_1}, \pi'_{i_1} \rangle \in \varrho, \ldots, \langle \pi_{i_m}, \pi'_{i_m} \rangle \in \varrho$,
then $\langle \rho, \pi'_1, \ldots, \pi'_n \rangle \in \xi$.

Suppose Σ is a signature and $\mu = \langle \beta, \varrho, \lambda, \xi \rangle$ is a Σ morph. We call β the basis set in μ, ϱ the re-entrancy relation in μ, λ the label function in μ, and ξ the relation extension in μ. We write $\mathcal{M}_\Sigma$ for the set of Σ morphs. Our Σ morphs are a straigthforward extension of abstract feature structures in the sense of (Moshier 1988).

34 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* $\pi \in \mathcal{A}^*$, *for each* $\langle \pi_1, \ldots, \pi_n \rangle \in \mathcal{A}^{**}$,

$\pi\langle \pi_1, \ldots, \pi_n \rangle$ *is an abbreviatory notation for* $\langle \pi\pi_1, \ldots, \pi\pi_n \rangle$.

35 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* $\mu = \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, *for each* $\pi \in \mathcal{A}^*$,

$\beta/\pi = \{\pi' \in \mathcal{A}^* \mid \pi\pi' \in \beta\}$,

$\varrho/\pi = \{\langle \pi_1, \pi_2 \rangle \in \mathcal{A}^* \times \mathcal{A}^* \mid \langle \pi\pi_1, \pi\pi_2 \rangle \in \varrho\}$,

$\lambda/\pi = \{\langle \pi', \sigma \rangle \in \mathcal{A}^* \times \mathcal{S} \mid \langle \pi\pi', \sigma \rangle \in \lambda\}$,

$\xi/\pi = \{\langle \rho, \pi_1, \ldots, \pi_n \rangle \in \mathcal{R} \times \overline{(\beta/\pi)}^* \mid \langle \rho, \pi\pi_1, \ldots, \pi\pi_n \rangle \in \xi\}$, *and*

$\mu/\pi = \langle \beta/\pi, \varrho/\pi, \lambda/\pi, \xi/\pi \rangle$.

If Σ is a signature, μ is a Σ morph and π is a Σ path then we call μ/π the π reduct of μ.

36 Proposition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* $\mu = \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, *for each* $\pi \in \mathcal{A}^*$,

if $\pi \in \beta$ *then* $\mu/\pi \in \mathcal{M}_\Sigma$.

37 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$,

I_Σ *is the set of total functions from* Var *to* $\overline{\mathcal{A}^*}$.

Let Σ be a signature. Then we call each member of I_Σ a Σ insertion.

38 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$,

T_Σ *is the smallest partial function from* $\overline{\mathcal{A}^*} \times \widehat{\mathcal{A}}$ *to* $\overline{\mathcal{A}^*}$ *such that,*

for each $\pi \in \mathcal{A}^*$, *for each* $\alpha \in \mathcal{A}$,

$\mathrm{T}_\Sigma(\pi, \alpha) = \pi\alpha$,

for each $\langle \pi_0, \ldots, \pi_n \rangle \in \mathcal{A}^{**}$,

$\mathrm{T}_\Sigma(\langle \pi_0, \ldots, \pi_n \rangle, \dagger) = \pi_0$,
$\mathrm{T}_\Sigma(\langle \pi_0, \ldots, \pi_n \rangle, \triangleright) = \langle \pi_1, \ldots, \pi_n \rangle$.

39 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* $\iota \in \mathrm{I}_\Sigma$,

Π^ι_Σ *is the smallest partial function from* $\mathcal{T}^\Sigma$ *to* $\overline{\mathcal{A}^*}$ *such that*

$\Pi^\iota_\Sigma(:) = \varepsilon$,
for each $v \in \mathsf{Var}$, $\Pi^\iota_\Sigma(v) = \iota(v)$, *and*
for each $\tau \in \mathcal{T}^\Sigma$, *for each* $\alpha \in \widehat{\mathcal{A}}$, $\Pi^\iota_\Sigma(\tau\alpha) = \mathrm{T}_\Sigma(\Pi^\iota_\Sigma(\tau), \alpha)$.

Suppose Σ is a signature, and ι is a Σ insertion. Then we call each Π^ι_Σ the path insertion function for ι in Σ.

40 Definition. *For each signature* Σ, *for each* $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$,

$\hat{\varrho}$ *is the smallest subset of* $\overline{\beta} \times \overline{\beta}$ *such that*

$\varrho \subseteq \hat{\varrho}$, *and*
for each $\pi_1 \in \beta, \ldots, \pi_n \in \beta$, *for each* $\pi'_1 \in \beta, \ldots, \pi'_n \in \beta$,

if $\langle \pi_1, \pi'_1 \rangle \in \varrho$, ..., *and* $\langle \pi_n, \pi'_n \rangle \in \varrho$
then $\langle \langle \pi_1, \ldots, \pi_n \rangle, \langle \pi'_1, \ldots, \pi'_n \rangle \rangle \in \hat{\varrho}$,

$\hat{\lambda}$ *is the total function from* $\overline{\beta}$ *to* $\widehat{\mathcal{S}}$ *such that*

for each $\pi \in \beta$, $\hat{\lambda}(\pi) = \lambda(\pi)$,
for each $\pi_1 \in \beta$, ..., *for each* $\pi_n \in \beta$,

$$\hat{\lambda}(\langle \pi_1, \dots, \pi_n \rangle) = \begin{cases} \textit{echain} & \textit{if } n = 0, \\ \textit{nechain} & \textit{if } n > 0. \end{cases}$$

41 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* $\iota \in \mathrm{I}_\Sigma$, Δ^ι_Σ *is the total function from* $\mathcal{D}^\Sigma$ *to* $\mathcal{M}_\Sigma$ *such that*

for each $\tau \in \mathcal{T}^\Sigma$, *for each* $\sigma \in \widehat{\mathcal{G}}$,

$$\Delta^\iota_\Sigma(\tau \sim \sigma) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \,\middle|\, \begin{array}{l} \Pi^\iota_\Sigma(\tau) \textit{ is defined, and,} \\ \textit{for some } \sigma' \in \widehat{\mathcal{S}}, \\ \quad \langle \Pi^\iota_\Sigma(\tau), \sigma' \rangle \in \hat{\lambda} \textit{ and } \sigma' \mathrel{\widehat{\sqsubseteq}} \sigma \end{array} \right\},$$

for each $\tau_1 \in \mathcal{T}^\Sigma$, *for each* $\tau_2 \in \mathcal{T}^\Sigma$,

$$\Delta^\iota_\Sigma(\tau_1 \approx \tau_2) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \,\middle|\, \begin{array}{l} \Pi^\iota_\Sigma(\tau_1) \textit{ is defined,} \\ \Pi^\iota_\Sigma(\tau_2) \textit{ is defined, and} \\ \langle \Pi^\iota_\Sigma(\tau_1), \Pi^\iota_\Sigma(\tau_2) \rangle \in \hat{\varrho} \end{array} \right\},$$

for each $\rho \in \mathcal{R}$, *for each* $v_1 \in \mathsf{Var}$, ..., *for each* $v_n \in \mathsf{Var}$,

$$\Delta^\iota_\Sigma(\rho(v_1, \dots, v_n)) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \,\middle|\, \langle \rho, \iota(v_1), \dots, \iota(v_n) \rangle \in \xi \right\},$$

for each $v \in \mathsf{Var}$, *for each* $\delta \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma(\exists v\, \delta) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \,\middle|\, \begin{array}{l} \textit{for some } \pi \in \overline{\beta}, \\ \quad \langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^{\iota[\frac{\pi}{v}]}_\Sigma(\delta) \end{array} \right\},$$

for each $v \in \mathsf{Var}$, *for each* $\delta \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma(\forall v\, \delta) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \,\middle|\, \begin{array}{l} \textit{for each } \pi \in \overline{\beta}, \\ \quad \langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^{\iota[\frac{\pi}{v}]}_\Sigma(\delta) \end{array} \right\},$$

for each $\delta \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma(\neg\delta) = \mathcal{M}_\Sigma \backslash \Delta^\iota_\Sigma(\delta),$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma([\delta_1 \wedge \delta_2]) = \Delta^\iota_\Sigma(\delta_1) \cap \Delta^\iota_\Sigma(\delta_2),$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma([\delta_1 \vee \delta_2]) = \Delta^\iota_\Sigma(\delta_1) \cup \Delta^\iota_\Sigma(\delta_2),$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma([\delta_1 \rightarrow \delta_2]) = (\mathcal{M}_\Sigma \backslash \Delta^\iota_\Sigma(\delta_1)) \cup \Delta^\iota_\Sigma(\delta_2), \textit{ and}$$

for each $\delta_1 \in \mathcal{D}^\Sigma$, *for each* $\delta_2 \in \mathcal{D}^\Sigma$,

$$\Delta^\iota_\Sigma([\delta_1 \leftrightarrow \delta_2]) = ((\mathcal{M}_\Sigma \backslash \Delta^\iota_\Sigma(\delta_1)) \cap (\mathcal{M}_\Sigma \backslash \Delta^\iota_\Sigma(\delta_2))) \cup (\Delta^\iota_\Sigma(\delta_1) \cap \Delta^\iota_\Sigma(\delta_2)).$$

Let Σ be a signature, μ a Σ morph, ι a Σ insertion and δ a Σ description. We call Δ^ι_Σ the morph satisfaction function in Σ, and say μ satisfies δ under ι in Σ if and only if $\Delta^\iota_\Sigma(\delta) = \mu$.

42 Proposition. *For each signature* Σ, *for each* $\iota_1 \in \mathrm{I}_\Sigma$, *for each* $\iota_2 \in \mathrm{I}_\Sigma$,

for each $\tau \in \mathcal{T}^\Sigma$,

if for each $v \in FV(\tau)$, $\iota_1(v) = \iota_2(v)$

then $\Pi^{\iota_1}_\Sigma(\tau)$ *is defined iff* $\Pi^{\iota_2}_\Sigma(\tau)$ *is defined, and*
if $\Pi^{\iota_1}_\Sigma(\tau)$ *is defined then* $\Pi^{\iota_1}_\Sigma(\tau) = \Pi^{\iota_2}_\Sigma(\tau)$, *and*

for each $\delta \in \mathcal{D}^\Sigma$,

if for each $v \in FV(\delta)$, $\iota_1(v) = \iota_2(v)$
then $\Delta^{\iota_1}_\Sigma(\delta) = \Delta^{\iota_2}_\Sigma(\delta)$.

Proof. *By induction on the length of* τ *and the complexity of* δ, *respectively.* ■

43 Definition. *For each signature* Σ,

M_Σ *is the total function from* $Pow(\mathcal{D}^\Sigma_0)$ *to* $Pow(\mathcal{M}_\Sigma)$ *such that for each* $\theta \subseteq \mathcal{D}^\Sigma_0$,

$$M_\Sigma(\theta) = \left\{ \langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma \;\middle|\; \begin{array}{l} \textit{for each } \pi \in \beta, \textit{ for each } \iota \in \mathrm{I}_\Sigma, \textit{ for each } \delta \in \theta, \\ \quad \langle \beta, \varrho, \lambda, \xi \rangle / \pi = \Delta^\iota_\Sigma(\delta) \end{array} \right\}.$$

Let Σ be a signature. We call M_Σ the morph admission function in Σ.

44 Definition. *For each signature* Σ,

o is a canonical object in Σ
iff o is a quintuple $\langle \beta, \varrho, \lambda, \xi, \eta \rangle$,
$\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, *and*
$\eta \in Quo(\varrho)$.[3]

Suppose that Σ is a signature. We write $\mathcal{C}_\Sigma$ for the set of canonical objects in Σ. Suppose further that $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$ and $\pi \in \beta$. We write $|\pi|_\varrho$ for the equivalence class of π in ϱ. Thus, we write $\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle$ for the canonical object $\langle \beta, \varrho, \lambda, \xi, \eta \rangle$, where η is the equivalence class of π in ϱ.

45 Definition. *For each signature* Σ, *for each* $\theta \in \mathcal{D}^\Sigma_0$, *for each* $\langle \langle \beta, \varrho, \lambda, \xi, |\pi_1|_\varrho \rangle, \ldots,$ $\langle \beta, \varrho, \lambda, \xi, |\pi_n|_\varrho \rangle \rangle \in (\mathbf{U}^\theta_\Sigma)^*$,

[3] $Quo(\varrho)$ is the quotient of ϱ, i.e., the set of equivalence classes of ϱ.

$\langle \beta, \varrho, \lambda, \xi, |\langle \pi_1, \ldots, \pi_n \rangle|_\varrho \rangle$ is an abbreviatory notation for $\langle \langle \beta, \varrho, \lambda, \xi, |\pi_1|_\varrho \rangle, \ldots, \langle \beta, \varrho, \lambda, \xi, |\pi_n|_\varrho \rangle \rangle$.

46 Definition. *For each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,*

$$\mathbf{U}_\Sigma^\theta = \left\{ \langle \beta, \varrho, \lambda, \xi, \eta \rangle \in \mathcal{C}_\Sigma \,\middle|\, \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta) \right\},$$

$$\mathbf{S}_\Sigma^\theta = \left\{ \langle u, \sigma \rangle \in \mathbf{U}_\Sigma^\theta \times \mathcal{S} \,\middle|\, \begin{array}{l} \text{for some } \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta), \\ \text{for some } \pi \in \beta, \\ \quad u = \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle, \text{ and} \\ \quad \sigma = \lambda(\pi) \end{array} \right\},$$

$\mathbf{A}_\Sigma^\theta$ is the total function from $\mathcal{A}$ to $Pow(\mathbf{U}_\Sigma^\theta \times \mathbf{U}_\Sigma^\theta)$ such that for each $\alpha \in \mathcal{A}$,

$$\mathbf{A}_\Sigma^\theta(\alpha) = \left\{ \langle u, u' \rangle \in \mathbf{U}_\Sigma^\theta \times \mathbf{U}_\Sigma^\theta \,\middle|\, \begin{array}{l} \text{for some } \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta), \\ \text{for some } \pi \in \beta, \\ \quad u = \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \text{ and} \\ \quad u' = \langle \beta, \varrho, \lambda, \xi, |\pi\alpha|_\varrho \rangle \end{array} \right\},$$

$\mathbf{R}_\Sigma^\theta$ is the total function from $\mathcal{R}$ to the powerset of $\overline{\mathbf{U}_\Sigma^\theta}^*$ such that for each $\rho \in \mathcal{R}$,

$$\mathbf{R}_\Sigma^\theta(\rho) = \left\{ \begin{array}{l} \langle u_1, \ldots, u_n \rangle \\ \in \overline{\mathbf{U}_\Sigma^\theta}^* \end{array} \,\middle|\, \begin{array}{l} \text{for some } \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta), \\ \text{for some } \langle \rho, \pi_1, \ldots, \pi_n \rangle \in \xi, \\ \quad u_1 = \langle \beta, \varrho, \lambda, \xi, |\pi_1|_\varrho \rangle, \ \ldots, \ u_n = \langle \beta, \varrho, \lambda, \xi, |\pi_n|_\varrho \rangle \end{array} \right\},$$

$$\mathbf{I}_\Sigma^\theta = \langle \mathbf{U}_\Sigma^\theta, \mathbf{S}_\Sigma^\theta, \mathbf{A}_\Sigma^\theta, \mathbf{R}_\Sigma^\theta \rangle.$$

47 Proposition. *For each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,*

$\mathbf{U}_\Sigma^\theta$ is a set,
$\mathbf{S}_\Sigma^\theta$ is a total function from $\mathbf{U}_\Sigma^\theta$ to $\mathcal{S}$,
$\mathbf{A}_\Sigma^\theta$ is a total function from $\mathcal{A}$ to the set of partial functions from $\mathbf{U}_\Sigma^\theta$ to $\mathbf{U}_\Sigma^\theta$,
$\mathbf{R}_\Sigma^\theta$ is a total function from $\mathcal{R}$ to the powerset of $\overline{\mathbf{U}_\Sigma^\theta}^*$, and
$\mathbf{I}_\Sigma^\theta$ is a Σ interpretation.

Let Σ be a signature and $\theta \subseteq \mathcal{D}_0^\Sigma$, We call $\mathbf{I}_\Sigma^\theta$ the canonical Σ interpretation of θ.

48 Proposition. *For each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each $\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \mathcal{C}_\Sigma$, for each total function ι from Var to $\overline{\beta/\pi}$, for each $\tau \in \mathcal{T}^\Sigma$,*

if $\Pi_\Sigma^\iota(\tau)$ is defined and $\Pi_\Sigma^\iota(\tau) \in \mathcal{A}^{**}$
then $\pi\Pi_\Sigma^\iota(\tau) \in \beta^*$.

Proof. *By arithmetic induction on the length of τ.* ■

49 Proposistion. *For each signature Σ, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,*

$\mathbf{I}^{\theta}_{\Sigma}$ *is a* $\langle\Sigma,\theta\rangle$ *model.*

Proof. *Throughout this proof, suppose that, for each signature* Σ, *for each* $\theta \subseteq \mathcal{D}^{\Sigma}_{0}$, *for each* $o = \langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle \in \mathbf{U}^{\theta}_{\Sigma}$, *for each total function* ι *from* Var *to* $\overline{\beta/\pi}$,

ass^{ι}_{o} *is the total function from* Var *to* $\overline{\mathbf{U}^{\theta}_{\Sigma}}$ *such that*

for each $v \in \mathsf{Var}$, $\mathrm{ass}^{\iota}_{o}(v) = \langle\beta,\varrho,\lambda,\xi,|\pi\iota(v)|_{\varrho}\rangle$.

Firstly, for each signature Σ, *for each* $\theta \subseteq \mathcal{D}^{\Sigma}_{0}$, *for each* $o = \langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle \in \mathbf{U}^{\theta}_{\Sigma}$, *for each* $\tau \in \mathcal{T}^{\Sigma}$, *for each total function* ι *from* Var *to* $\overline{\beta/\pi}$,

$T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle)$ *is defined iff* $\Pi^{\iota}_{\Sigma}(\tau)$ *is defined and* $\pi\Pi^{\iota}_{\Sigma}(\tau) \in \overline{\beta}$, *and*
if $T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle)$ *is defined*
then $T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle) = \langle\beta,\varrho,\lambda,\xi,|\pi\Pi^{\iota}_{\Sigma}(\tau)|_{\varrho}\rangle$.
by proposition 48 and arithmetic induction on the length of τ

Secondly, for each signature $\Sigma = \langle\mathcal{G},\sqsubseteq,\mathcal{S},\mathcal{A},\mathcal{F},\mathcal{R},\mathcal{AR}\rangle$, *for each* $\theta \subseteq \mathcal{D}^{\Sigma}_{0}$, *for each* $o = \langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle \in \mathbf{U}^{\theta}_{\Sigma}$, *for each total function* ι *from* Var *to* $\overline{\beta/\pi}$,

for each $\tau \in \mathcal{T}^{\Sigma}$, *for each* $\sigma \in \mathcal{G}$,

$$\begin{aligned}
&\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle \in D^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau \sim \sigma)\\
\Longleftrightarrow\ & T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle) \text{ is defined and } \widehat{\mathbf{S}^{\theta}_{\Sigma}}(T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle)) \mathrel{\widehat{\sqsubseteq}} \sigma\\
\Longleftrightarrow\ & \Pi^{\iota}_{\Sigma}(\tau) \text{ is defined, } \pi\Pi^{\iota}_{\Sigma}(\tau) \in \overline{\beta}, \text{ and } \widehat{\mathbf{S}^{\theta}_{\Sigma}}(\langle\beta,\varrho,\lambda,\xi,|\pi\Pi^{\iota}_{\Sigma}(\tau)|_{\varrho}\rangle) \mathrel{\widehat{\sqsubseteq}} \sigma\\
\Longleftrightarrow\ & \Pi^{\iota}_{\Sigma}(\tau) \text{ is defined, and } \widehat{\lambda/\pi}(\Pi^{\iota}_{\Sigma}(\tau)) \mathrel{\widehat{\sqsubseteq}} \sigma\\
\Longleftrightarrow\ & \langle\beta,\varrho,\lambda,\xi\rangle/\pi \in \Delta^{\iota}_{\Sigma}(\tau \sim \sigma),
\end{aligned}$$

for each $\tau_1 \in \mathcal{T}^{\Sigma}$, *for each* $\tau_2 \in \mathcal{T}^{\Sigma}$,

$$\begin{aligned}
&\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle \in D^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau_1 \approx \tau_2)\\
\Longleftrightarrow\ & T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau_1)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle) \text{ is defined, } T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau_2)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle) \text{ is defined, and}\\
& T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau_1)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle) = T^{\mathrm{ass}^{\iota}_{o}}_{\mathbf{I}^{\theta}_{\Sigma}}(\tau_2)(\langle\beta,\varrho,\lambda,\xi,|\pi|_{\varrho}\rangle)\\
\Longleftrightarrow\ & \Pi^{\iota}_{\Sigma}(\tau_1) \text{ is defined, } \pi\Pi^{\iota}_{\Sigma}(\tau_1) \in \overline{\beta}, \Pi^{\iota}_{\Sigma}(\tau_2) \text{ is defined, } \pi\Pi^{\iota}_{\Sigma}(\tau_2) \in \overline{\beta}, \text{ and}\\
& \langle\beta,\varrho,\lambda,\xi,|\pi\Pi^{\iota}_{\Sigma}(\tau_1)|_{\varrho}\rangle = \langle\beta,\varrho,\lambda,\xi,|\pi\Pi^{\iota}_{\Sigma}(\tau_2)|_{\varrho}\rangle\\
\Longleftrightarrow\ & \Pi^{\iota}_{\Sigma}(\tau_1) \text{ is defined, } \Pi^{\iota}_{\Sigma}(\tau_2) \text{ is defined, and } \langle\Pi^{\iota}_{\Sigma}(\tau_1),\Pi^{\iota}_{\Sigma}(\tau_2)\rangle \in \widehat{\varrho/\pi}\\
\Longleftrightarrow\ & \langle\beta,\varrho,\lambda,\xi\rangle/\pi \in \Delta^{\iota}_{\Sigma}(\tau_1 \approx \tau_2),
\end{aligned}$$

for each $\rho \in \mathcal{R}$, *for each* $x_1 \in \mathsf{Var}$, $\ldots$, *for each* $x_{\mathcal{AR}(\rho)} \in \mathsf{Var}$,

$$\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in D^{\text{ass}^\iota_o}_{\mathbf{I}^\theta_\Sigma}(\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)}))$$
$$\iff \langle \text{ass}^\iota_o(x_1), \ldots, \text{ass}^\iota_o(x_{\mathcal{AR}(\rho)}) \rangle \in \mathbf{R}^\theta_\Sigma(\rho)$$
$$\iff \langle \langle \beta, \varrho, \lambda, \xi, |\pi\iota(x_1)|_\varrho \rangle, \ldots, \langle \beta, \varrho, \lambda, \xi, |\pi\iota(x_{\mathcal{AR}(\rho)})|_\varrho \rangle \rangle \in \mathbf{R}^\theta_\Sigma(\rho)$$
$$\iff \langle \rho, \pi\iota(x_1), \ldots, \pi\iota(x_{\mathcal{AR}(\rho)}) \rangle \in \xi$$
$$\iff \langle \rho, \iota(x_1), \ldots, \iota(x_{\mathcal{AR}(\rho)}) \rangle \in \xi/\pi$$
$$\iff \langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta^\iota_\Sigma(\rho(x_1, \ldots, x_{\mathcal{AR}(\rho)})).$$

Thus, for each signature Σ, *for each* $\theta \subseteq \mathcal{D}^\Sigma_0$, *for each* $o = \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \mathbf{U}^\theta_\Sigma$, *for each* $\delta \in \mathcal{D}^\Sigma$, *for each total function* ι *from* Var *to* $\overline{\beta/\pi}$,

$$\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in D^{\text{ass}^\iota_o}_{\mathbf{I}^\theta_\Sigma}(\delta) \text{ iff } \langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta^\iota_\Sigma(\delta).$$

by arithmetic induction on the complexity of δ

(since, for each $\pi' \in \overline{\beta/\pi}$, $\text{ass}^{\iota\frac{\langle \beta,\varrho,\lambda,\xi,|\pi\pi'|_\varrho \rangle}{v}}_o = \text{ass}^{\iota\frac{\pi'}{v}}_o$ *)*

Thus, for each signature Σ, *for each* $\theta \subseteq \mathcal{D}^\Sigma_0$, *for each* $\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \mathcal{C}_\Sigma$,

$\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \mathbf{U}^\theta_\Sigma$,

$\Longrightarrow \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \mathbf{U}^\theta_\Sigma$, $\langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta)$ *and* $\pi \in \beta$

$\Longrightarrow \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \mathbf{U}^\theta_\Sigma$,
for each $\delta \in \theta$, *for each total function* ι *from* Var *to* $\overline{\beta/\pi}$,

$\langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta^\iota_\Sigma(\delta)$

$\Longrightarrow$ *for each* $\delta \in \theta$, *for each total function* ι *from* Var *to* $\overline{\beta/\pi}$,

$\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in D^{\text{ass}^\iota_o}_{\mathbf{I}^\theta_\Sigma}(\delta)$

$\Longrightarrow$ *for each* $\delta \in \theta$, *for some* $\text{ass} \in Ass_{\mathbf{I}^\theta_\Sigma}$,

$\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in D^{\text{ass}}_{\mathbf{I}^\theta_\Sigma}(\delta)$

$\Longrightarrow$ *for each* $\delta \in \theta$, *for each* $\text{ass} \in Ass_{\mathbf{I}^\theta_\Sigma}$,

$\langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in D^{\text{ass}}_{\mathbf{I}^\theta_\Sigma}(\delta)$ *by corollary 17*

$\Longrightarrow \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle \in \Theta_{\mathbf{I}^\theta_\Sigma}(\theta)$.

■

50 Definition. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$,

A_I *is the binary relation on* $U \times \mathcal{M}_\Sigma$ *such that, for each* $u \in U$, *for each* $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, $\langle u, \langle \beta, \varrho, \lambda, \xi \rangle \rangle \in \mathsf{A}_I$ *iff*

$$\beta = \left\{ \pi \in \mathcal{A}^* \,\middle|\, T_I(\pi, u) \text{ is defined} \right\},$$

$$\varrho = \left\{ \langle \pi_1, \pi_2 \rangle \in \mathcal{A}^* \times \mathcal{A}^* \,\middle|\, \begin{array}{l} T_I(\pi_1, u) \text{ is defined,} \\ T_I(\pi_2, u) \text{ is defined, and} \\ T_I(\pi_1, u) = T_I(\pi_2, u) \end{array} \right\},$$

$$\lambda = \left\{ \langle \pi, \sigma \rangle \in \mathcal{A}^* \times \mathcal{S} \,\middle|\, \begin{array}{l} T_I(\pi, u) \text{ is defined,} \\ S(T_I(\pi, u)) = \sigma \end{array} \right\},$$

$$\xi = \left\{ \langle \rho, \pi_1, \ldots, \pi_n \rangle \in \mathcal{R} \times (\overline{\mathcal{A}^*})^* \,\middle|\, \begin{array}{l} T_I(\pi_1, u) \text{ is defined, } \ldots, \\ T_I(\pi_n, u) \text{ is defined, and} \\ \langle T_I(\pi_1, u), \ldots, T_I(\pi_n, u) \rangle \in R(\rho) \end{array} \right\}.$$

51 Proposition. *For each signature Σ, for each Σ interpretation $I = \langle U, S, A, R \rangle$,*

A_I *is a total function from U to $\mathcal{M}_\Sigma$.*

52 Proposition. *For each signature Σ, for each Σ interpretation $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$, for each Σ interpretation $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$, for each $o_1 \in U_1$, for each $o_2 \in U_2$,*

$\mathsf{A}_{I_1}(o_1) = \mathsf{A}_{I_2}(o_2)$
iff $\langle o_1, I_1 \rangle$ and $\langle o_2, I_2 \rangle$ are congruent in Σ.

Proof. *Firstly, for each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each Σ interpretation $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$, for each Σ interpretation $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$, for each $o_1 \in U_1$, for each $o_2 \in U_2$,*

$\mathsf{A}_{I_1}(o_1) = \mathsf{A}_{I_2}(o_2)$

$$\Longrightarrow \left\{ \langle o_1', o_2' \rangle \in \overline{U_1} \times \overline{U_2} \,\middle|\, \begin{array}{l} \text{for some } \pi \in \overline{\mathcal{A}^*}, \\ \quad T_{I_1}(\pi, o_1) \text{ is defined,} \\ \quad T_{I_2}(\pi, o_2) \text{ is defined,} \\ \quad o_1' = T_{I_1}(\pi, o_1), \text{ and} \\ \quad o_2' = T_{I_2}(\pi, o_2) \end{array} \right\}$$

is a congruence from $\langle o_1, I_1 \rangle$ to $\langle o_2, I_2 \rangle$ in Σ

$\Longrightarrow$ *$\langle o_1, I_1 \rangle$ and $\langle o_2, I_2 \rangle$ are congruent in Σ.*

Secondly, for each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each Σ interpretation $I_1 = \langle U_1, S_1, A_1, R_1 \rangle$, for each Σ interpretation $I_2 = \langle U_2, S_2, A_2, R_2 \rangle$, for each $o_1 \in U_1$, for each $o_2 \in U_2$,

$\langle o_1, I_1 \rangle$ and $\langle o_2, I_2 \rangle$ are congruent in Σ

$\Longrightarrow$ *for some f, f is a congruence from $\langle o_1, I_1 \rangle$ to $\langle o_2, I_2 \rangle$*

$\Longrightarrow$ *for some congruence f from $\langle o_1, I_1\rangle$ to $\langle o_2, I_2\rangle$, for each $\pi \in \mathcal{A}^*$,*
$T_{I_1}(\pi, o_1)$ is defined iff $T_{I_2}(\pi, o_2)$ is defined, and
if $T_{I_1}(\pi, o_1)$ is defined then $f(T_{I_1}(\pi, o_1)) = T_{I_2}(\pi, o_2)$
by induction on the length of π

$\Longrightarrow \mathsf{A}_{I_1}(o_1) = \mathsf{A}_{I_2}(o_2)$.

■

53 Lemma. *For each signature Σ, for each $\langle \beta, \varrho, \lambda, \xi\rangle \in \mathcal{M}_\Sigma$,*

for each $\tau \in \mathcal{T}^\Sigma$, for each $\iota_1 \in \mathrm{I}_\Sigma$, for each $\iota_2 \in \mathrm{I}_\Sigma$,

if for each $v \in \mathsf{Var}$, $\iota_1(v) \in \overline{\beta}$ and $\iota_2(v) \in \overline{\beta}$, and
for each $v \in FV(\tau)$, $\langle \iota_1(v), \iota_2(v)\rangle \in \hat{\varrho}$

then $\Pi^{\iota_1}_\Sigma(\tau)$ is defined and $\Pi^{\iota_1}_\Sigma(\tau) \in \overline{\beta}$ iff $\Pi^{\iota_2}_\Sigma(\tau)$ is defined and $\Pi^{\iota_2}_\Sigma(\tau) \in \overline{\beta}$, and
if $\Pi^{\iota_1}_\Sigma(\tau)$ is defined and $\Pi^{\iota_1}_\Sigma(\tau) \in \overline{\beta}$ then $\langle \Pi^{\iota_1}_\Sigma(\tau), \Pi^{\iota_2}_\Sigma(\tau)\rangle \in \hat{\varrho}$, and

for each $\delta \in \mathcal{D}^\Sigma$, for each $\iota_1 \in \mathrm{I}_\Sigma$, for each $\iota_2 \in \mathrm{I}_\Sigma$,

if for each $v \in \mathsf{Var}$, $\iota_1(v) \in \overline{\beta}$ and $\iota_2(v) \in \overline{\beta}$, and
for each $v \in FV(\delta)$, $\langle \iota_1(v), \iota_2(v)\rangle \in \hat{\varrho}$

then $\langle \beta, \varrho, \lambda, \xi\rangle \in \Delta^{\iota_1}_\Sigma(\delta)$ iff $\langle \beta, \varrho, \lambda, \xi\rangle \in \Delta^{\iota_2}_\Sigma(\delta)$.

Proof. *By proposition 48 and induction on the length of τ, and by induction on the complexity of δ, respectively.* ■

54 Definition. *For each signature Σ, for each Σ interpretation $I = \langle U, S, A, R\rangle$, for each $o \in U$, for each $\iota \in \mathrm{I}_\Sigma$,*

$$\mathrm{ass}^\iota_{o,I} = \left\{ \langle v, o'\rangle \in \mathsf{Var} \times \overline{U} \;\middle|\; \begin{array}{l} T_I(\Pi^\iota_\Sigma(v), o) \textit{ is defined, and} \\ T_I(\Pi^\iota_\Sigma(v), o) = o' \end{array} \right\}.$$

55 Proposition. *For each signature Σ, for each Σ interpretation $I = \langle U, S, A, R\rangle$, for each $o \in U$, for each $\langle \beta, \varrho, \lambda, \xi\rangle \in \mathcal{M}_\Sigma$, for each $\iota \in \mathrm{I}_\Sigma$,*

if for each $v \in \mathsf{Var}$, $\iota(v) \in \overline{\beta}$, and
$\mathsf{A}_I(o) = \langle \beta, \varrho, \lambda, \xi\rangle$

then $\mathrm{ass}^\iota_{o,I} \in Ass_I$.

56 Lemma. *For each signature Σ, for each Σ interpretation $I = \langle U, S, A, R\rangle$, for each $o \in U$, for each $\langle \beta, \varrho, \lambda, \xi\rangle \in \mathcal{M}_\Sigma$, for each $\tau \in \mathcal{T}^\Sigma$, for each $\iota \in \mathrm{I}_\Sigma$,* *if for each* $v \in \mathsf{Var}$, $\iota(v) \in \overline{\beta}$, *and*

$\mathsf{A}_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$

then $T_I^{\mathrm{ass}^\iota_{o,I}}(\tau)(o)$ *is defined iff* $T_I(\Pi^\iota_\Sigma(\tau), o)$ *is defined, and*

if $T_I^{\mathrm{ass}^\iota_{o,I}}(\tau)(o)$ *is defined then* $T_I^{\mathrm{ass}^\iota_{o,I}}(\tau)(o) = T_I(\Pi^\iota_\Sigma(\tau), o)$.

Proof. *By proposition 48 and induction on the length of* τ. ■

57 Proposition. *For each signature* Σ, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $o \in U$, *for each* $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, *for each* $\delta \in \mathcal{D}^\Sigma$, *for each* $\iota \in \mathrm{I}_\Sigma$,

if for each $v \in \mathsf{Var}$, $\iota(v) \in \overline{\beta}$, *and*

$\mathsf{A}_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$

then $o \in D_I^{\mathrm{ass}^\iota_{o,I}}(\delta)$ *iff* $\langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^\iota_\Sigma(\delta)$.

Proof. *For each signature* Σ, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $o \in U$, *for each* $o' \in \overline{Co^o_I}$, *let* $\#(o')$ *be a* $\pi \in \overline{\beta}$ *such that*

$o' = T_I(\pi, o)$.

We can then show that for each signature Σ, *for each* Σ *interpretation* $I = \langle U, S, A, R \rangle$, *for each* $o \in U$, *for each* $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$, *for each* $\iota \in \mathrm{I}_\Sigma$,

if for each $v \in \mathsf{Var}$, $\iota(v) \in \overline{\beta}$, *and*

$\mathsf{A}_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$

then for each $o' \in \overline{Co^o_I}$, *for each* $v \in \mathsf{Var}$, $\mathrm{ass}^\iota_{o,I}\frac{o'}{v} = \mathrm{ass}^{\iota\frac{\#(o')}{v}}_{o,I}$, *and*

for each $\pi \in \overline{\beta}$, *for each* $o' \in \overline{Co^o_I}$,

$$
\begin{aligned}
& o' = T_I(\pi, o) \\
& \Longrightarrow T_I(\#(o'), o) = T_I(\pi, o) \\
& \Longrightarrow \langle \#(o'), \pi \rangle \in \widehat{\varrho} \\
& \Longrightarrow \text{for each } \delta \in \mathcal{D}^\Sigma, \\
& \qquad \langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^{\iota\frac{\#(o')}{v}}_\Sigma(\delta) \text{ iff } \langle \beta, \varrho, \lambda, \xi \rangle \in \Delta^{\iota\frac{\pi}{v}}_\Sigma(\delta). \qquad \text{by lemma 53}
\end{aligned}
$$

Using this result, the proposition follows by lemma 56 and induction on the complexity of δ. ■

58 Lemma. *For each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each interpretation* $I = \langle U, S, A, R \rangle$, *for each* $o \in U$, *for each* $\pi \in \mathcal{A}^*$, *for each* $\pi' \in \mathcal{A}^*$,

$T_I(\pi\pi', o)$ *is defined iff* $T_I(\pi', T_I(\pi, o))$ *is defined, and*

if $T_I(\pi\pi', o)$ *is defined then* $T_I(\pi\pi', o) = T_I(\pi', T_I(\pi, o))$.

Proof. *By induction on the length of π'.* ■

59 Lemma. *For each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\pi \in \mathcal{A}^*$, for each $\pi' \in \mathcal{A}^{**}$,*

$T_I(\pi\pi', o)$ is defined iff $T_I(\pi', T_I(\pi, o))$ is defined, and
if $T_I(\pi\pi', o)$ is defined then $T_I(\pi\pi', o) = T_I(\pi', T_I(\pi, o))$.

Proof. *Follows from lemma 58.* ■

60 Proposition. *For each signature $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, for each Σ interpretation $I = \langle U, S, A, R \rangle$, for each $o \in U$, for each $\pi \in \mathcal{A}^*$,*

if $T_I(\pi, o)$ is defined then $\mathsf{A}_I(T_I(\pi, o)) = \mathsf{A}_I(o)/\pi$.

Proof. *Uses lemma 58 and lemma 59.* ■

61 Proposition. *For each signature Σ, for each Σ interpretation $I = \langle U, S, A, R \rangle$, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,*

if I is a $\langle \Sigma, \theta \rangle$ model
then for each $o \in U$, $\mathsf{A}_I(o) \in M_\Sigma(\theta)$.

Proof. *For each signature Σ, for each Σ interpretation $I = \langle U, S, A, R \rangle$, for each $\theta \subseteq \mathcal{D}_0^\Sigma$,*

I is a $\langle \Sigma, \theta \rangle$ model

$\Longrightarrow$ *for each $o \in U$, for each $\langle \beta, \varrho, \lambda, \xi \rangle \in \mathcal{M}_\Sigma$,*

$\langle \beta, \varrho, \lambda, \xi \rangle = \mathsf{A}_I(o)$

$\Longrightarrow$ *for each $\pi \in \beta$,*
$\langle \beta, \varrho, \lambda, \xi \rangle/\pi = \mathsf{A}_I(T_I(\pi, o))$ *by proposition 60*

$\Longrightarrow$ *for each $\pi \in \beta$, for each $\mathrm{ass} \in Ass_I$, for each $\delta \in \theta$,*
$\langle \beta, \varrho, \lambda, \xi \rangle/\pi = \mathsf{A}_I(T_I(\pi, o))$ *and* $T_I(\pi, o) \in D_I^{\mathrm{ass}}(\delta)$

$\Longrightarrow$ *for each $\pi \in \beta$, for each total function $\iota\colon \mathsf{Var} \to \overline{\beta/\pi}$, for each $\delta \in \theta$,*
$\langle \beta, \varrho, \lambda, \xi \rangle/\pi = \mathsf{A}_I(T_I(\pi, o))$ *and* $T_I(\pi, o) \in D_I^{\mathrm{ass}^\iota_{T_I(\pi,o),I}}(\delta)$
by proposition 55

$\Longrightarrow$ *for each $\pi \in \beta$, for each total function $\iota\colon \mathsf{Var} \to \overline{\beta/\pi}$, for each $\delta \in \theta$,*
$\langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta_\Sigma^\iota(\delta)$ *by proposition 57*

$\Longrightarrow$ *for each $\pi \in \beta$, for some $\iota \in \mathrm{I}_\Sigma$, for each $\delta \in \theta$,*
$\langle \beta, \varrho, \lambda, \xi \rangle/\pi \in \Delta_\Sigma^\iota(\delta)$ *since $\beta/\pi \neq \emptyset$*

$\Longrightarrow$ *for each* $\pi \in \beta$, *for each* $\iota \in \mathrm{I}_\Sigma$, *for each* $\delta \in \theta$,
$\langle \beta, \varrho, \lambda, \xi \rangle / \pi \in \Delta^\iota_\Sigma(\delta)$ *by proposition 42*

$\Longrightarrow \langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta)$

$\Longrightarrow$ *for each* $o \in U$, $\mathsf{A}_I(o) \in M_\Sigma(\theta)$. *by proposition 51*

■

62 Proposition. *For each signature* Σ, *for each* $\theta \subseteq \mathcal{D}^\Sigma_0$, *for each* $\langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta)$,

$\mathsf{A}_{\mathbf{I}^\theta_\Sigma}(\langle \beta, \varrho, \lambda, \xi, |\varepsilon|_\varrho \rangle) = \langle \beta, \varrho, \lambda, \xi \rangle$.

Proof. *Follows from the observation that, for each signature* $\Sigma = \langle \mathcal{G}, \sqsubseteq, \mathcal{S}, \mathcal{A}, \mathcal{F}, \mathcal{R}, \mathcal{AR} \rangle$, *for each* $\theta \subseteq \mathcal{D}^\Sigma_0$, *for each* $\langle \beta, \varrho, \lambda, \xi, |\varepsilon|_\varrho \rangle \in \mathbf{U}^\theta_\Sigma$, *for each* $\pi \in \mathcal{A}^*$,

$\pi \in \beta$ *iff* $T_{\mathbf{I}^\theta_\Sigma}(\pi, \langle \beta, \varrho, \lambda, \xi, |\varepsilon|_\varrho \rangle)$ *is defined, and*
if $\pi \in \beta$ *then* $T_{\mathbf{I}^\theta_\Sigma}(\pi, \langle \beta, \varrho, \lambda, \xi, |\varepsilon|_\varrho \rangle) = \langle \beta, \varrho, \lambda, \xi, |\pi|_\varrho \rangle$. *by induction on the length of* π

■

63 Proposition. *For each signature* Σ, *for each* $\theta \subseteq \mathcal{D}^\Sigma_0$,

$\mathbf{I}^\theta_\Sigma$ *is an exhaustive* $\langle \Sigma, \theta \rangle$ *model.*

Proof. *For each signature* Σ, *for each* $\theta \subseteq \mathcal{D}^\Sigma_0$,

$\mathbf{I}^\theta_\Sigma$ *is a* $\langle \Sigma, \theta \rangle$ *model, and* *by proposition 49*
for each Σ *interpretation* $I = \langle U, S, A, R \rangle$,

I *is a* $\langle \Sigma, \theta \rangle$ *model*

$\Longrightarrow$ *for each* $o \in U$,
$\mathsf{A}_I(o) \in M_\Sigma(\theta)$ *by proposition 61*

$\Longrightarrow$ *for each* $o \in U$, *for some* $\langle \beta, \varrho, \lambda, \xi \rangle \in M_\Sigma(\theta)$,
$\mathsf{A}_I(o) = \langle \beta, \varrho, \lambda, \xi \rangle$, *and*
$\mathsf{A}_{\mathbf{I}^\theta_\Sigma}(\langle \beta, \varrho, \lambda, \xi, |\varepsilon|_\varrho \rangle) = \langle \beta, \varrho, \lambda, \xi \rangle$ *by proposition 62*

$\Longrightarrow$ *for each* $o \in U$, *for some* $o' \in \mathbf{U}^\theta_\Sigma$,
$\langle o', \mathbf{I}^\theta_\Sigma \rangle$ *and* $\langle o, I \rangle$ *are congruent in* Σ *by proposition 52*

$\Longrightarrow \mathbf{I}^\theta_\Sigma$ *simulates* I *in* Σ

$\Longrightarrow \mathbf{I}^\theta_\Sigma$ *is an exhaustive* $\langle \Sigma, \theta \rangle$ *model.* *by theorem 32*

■

64 Theorem. *For each signature* Σ*, for each theory* θ*, there exists a* Σ *interpretation* I *such that*

I *is an exhaustive* $\langle \Sigma, \theta \rangle$ *model.*

An immediate corollary of Theorem 64 and the definition of an exhaustive model 22 is that if a grammar has a non-empty model then it has a non-empty exhaustive model.

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