Frank Richter

Algorithmic Semantics

Syntax and Semantics of First Order Logic

Syntax Let Var, Const, Func and Rel be at most countably infinite and pairwise disjoint sets of symbols. Each symbol in Func and in Rel is assigned a natural number called the *arity* of the symbol.
To simplify the formulations in the definitions of the syntax and semantics, we will henceforth assume that the sets Var, Const, Func and Rel are fixed.

Definition 1 Terms

i.) *For every* $v \in \mathsf{Var}$*,* v *is a term.*

ii.) *For every* $c \in \mathsf{Const}$*,* c *is a term.*

iii.) *For every term* t_1*, ..., for every term* t_n*, for every* n*-ary function symbol* F*,* $F \in \mathsf{Func}$*,* $F(t_1, \ldots, t_n)$ *is a term.*

iv.) *Only that which can be generated by the clauses i.)–iii.) in a finite number of steps is a term.*

Definition 2 Formulae

i.) *For every term* t_1*, for every term* t_2*,* $t_1 \equiv t_2$ *is a formula.*

ii.) *For every term* t_1*, ..., for every term* t_n*, for every* n*-ary relation symbol* R*,* $R \in \mathsf{Rel}$*,* $R(t_1, \ldots, t_n)$ *is a formula.*

iii.) *For every formula* ϕ*,* $\neg\phi$ *is a formula.*

iv.) *For every formula* ϕ_1*, for every formula* ϕ_2*,* $(\phi_1 \wedge \phi_2)$ *is a formula.*

v.) *For every formula* ϕ_1*, for every formula* ϕ_2*,* $(\phi_1 \vee \phi_2)$ *is a formula.*

vi.) *For every formula* ϕ_1*, for every formula* ϕ_2*,* $(\phi_1 \rightarrow \phi_2)$ *is a formula.*

vii.) *For every formula* ϕ_1*, for every formula* ϕ_2*,* $(\phi_1 \leftrightarrow \phi_2)$ *is a formula.*

viii.) *For every* $v \in \mathsf{Var}$*, for every formula* ϕ*,* $\forall v\, \phi$ *is a formula.*

ix.) *For every* $v \in \mathsf{Var}$*, for every formula* ϕ*,* $\exists v\, \phi$ *is a formula.*

x.) *Only that which can be generated by the clauses i.)–ix.) in a finite number of steps is a formula.*

On the basis of the syntactic form of first order formulae we can say what it means for a variable to occur *free* in an expression.
To make this precise, we first define a function, var, which assigns to each term the set of variables in it. Then we define a function, free, which assigns to each formula ϕ the set of variables which occur free in ϕ.

Definition 3 var
var *is the total function from the set of terms to the powerset of* Var *such that:*

i.) *For every* $v \in \mathsf{Var}$, $\mathsf{var}(v) = \{v\}$.

ii.) *For every* $c \in \mathsf{Const}$, $\mathsf{var}(c) = \emptyset$.

iii.) *For every term* t_1, ..., *for every term* t_n, *for every* n*-ary function symbol* F, $F \in \mathsf{Func}$,
$\mathsf{var}\left(F\left(t_1, \ldots, t_n\right)\right) = \mathsf{var}(t_1) \cup \ldots \cup \mathsf{var}(t_n)$.

Definition 4 free
free *is the total function from the set of formulae to the powerset of* Var *such that:*

i.) *For every term* t_1, *for every term* t_2,
$\mathsf{free}\left(t_1 \equiv t_2\right) = \mathsf{var}(t_1) \cup \mathsf{var}(t_2)$.

ii.) *For every term* t_1, ..., *for every term* t_n, *for every* n*-ary relation symbol* R, $R \in \mathsf{Rel}$,
$\mathsf{free}\left(R\left(t_1, \ldots, t_n\right)\right) = \mathsf{var}(t_1) \cup \ldots \cup \mathsf{var}(t_n)$.

iii.) *For every formula* ϕ, $\mathsf{free}\left(\neg\phi\right) = \mathsf{free}\left(\phi\right)$.

iv.) *For every formula* ϕ_1, *for every formula* ϕ_2, *for* $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,
$\mathsf{free}\left(\left(\phi_1 * \phi_2\right)\right) = \mathsf{free}\left(\phi_1\right) \cup \mathsf{free}\left(\phi_2\right)$.

v.) *For every* $v \in \mathsf{Var}$, *for every formula* ϕ,
$\mathsf{free}\left(\forall v\ \phi\right) = \mathsf{free}\left(\phi\right) \setminus \{v\}$.

vi.) *For every* $v \in \mathsf{Var}$, *for every formula* ϕ,
$\mathsf{free}\left(\exists v\ \phi\right) = \mathsf{free}\left(\phi\right) \setminus \{v\}$.

For each formula ϕ, $\mathsf{free}\left(\phi\right)$ is the set of variables which occur free in ϕ.
For each formula of the form $\mathsf{Q}v\ \phi$ (with Q a quantifier), we call ϕ the *scope* of the quantifier Q. We say that in a formula $\mathsf{Q}v\ \phi$ the quantifier Q binds all instances of the variable v which occur free in ϕ.

Semantics Let D be a set of objects, called the *domain* of our first order terms and formulae, and I a total function from $\mathsf{Const} \cup \mathsf{Func} \cup \mathsf{Rel}$ to $\mathsf{D} \cup Pow\left(\bigcup_{n \in \mathbb{N}} \mathsf{D_1} \times \ldots \times \mathsf{D_n}\right)$ which assigns to each $c \in \mathsf{Const}$ an element of D; to each m-ary function symbol $F \in \mathsf{Func}$ a function from D^m to D; and to each n-ary relation symbol $R \in \mathsf{Rel}$ a subset of $\mathsf{D_1} \times \ldots \times \mathsf{D_n}$.
Let $\mathsf{M} = \langle \mathsf{D}, \mathsf{I} \rangle$. We will call each M a *model*.
Let g be a function in $\mathsf{D}^{\mathsf{Var}}$ which assigns to each variable in Var an object in the domain D. We call each g an *assignment function*.

Definition 5 Term Interpretation
Let M *be a model and* g *an assignment function.*

i.) *For every* $v \in \mathsf{Var}$, $[\![v]\!]^{\mathsf{M},g} = g(v)$.

ii.) *For every* $c \in \mathsf{Const}$, $[\![c]\!]^{\mathsf{M},g} = \mathsf{I}(c)$.

iii.) *For every term* t_1, *. . . , for every term* t_n, *for every* n*-ary function symbol* F, $F \in \mathsf{Func}$,

$[\![F(t_1, \ldots, t_n)]\!]^{\mathsf{M},g} = \mathsf{I}(F)\left(\langle [\![t_1]\!]^{\mathsf{M},g}, \ldots [\![t_n]\!]^{\mathsf{M},g}\rangle\right)$.

Assume that $d \in \mathsf{D}$ and v is a variable. In what follows we will use the notation $g\overset{d}{v}$ for the assignment function g' which differs from the assignment function g in the following way:

For each $x \in \mathsf{Var}$, $g\overset{d}{v}(x) = \begin{cases} d & \text{if } x = v, \text{ and} \\ g(x) & \text{otherwise.} \end{cases}$

Definition 6 Formula Validation
Let $\mathsf{M} = \langle \mathsf{D}, \mathsf{I} \rangle$ *be a model and* g *an assignment function.*

i.) *For every term* t_1, *for every term* t_2,

$\mathsf{V}^{\mathsf{M},g}(t_1 \equiv t_2) = 1$ *iff* $[\![t_1]\!]^{\mathsf{M},g} = [\![t_2]\!]^{\mathsf{M},g}$.

ii.) *For every term* t_1, *. . . , for every term* t_n, *for every* n*-ary relation symbol* R, $R \in \mathsf{Rel}$,

$\mathsf{V}^{\mathsf{M},g}\left(R\left(t_1, \ldots t_n\right)\right) = 1$ *iff* $\langle [\![t_1]\!]^{\mathsf{M},g}, \ldots [\![t_n]\!]^{\mathsf{M},g}\rangle \in \mathsf{I}(R)$.

iii.) *For every formula* ϕ,

$\mathsf{V}^{\mathsf{M},g}(\neg\phi) = 1$ *iff* $\mathsf{V}^{\mathsf{M},g}(\phi) = 0$.

iv.) *For every formula* ϕ_1, *for every formula* ϕ_2,

$\mathsf{V}^{\mathsf{M},g}\left((\phi_1 \wedge \phi_2)\right) = 1$ *iff* $\mathsf{V}^{\mathsf{M},g}\left(\phi_1\right) = 1$ *and* $\mathsf{V}^{\mathsf{M},g}\left(\phi_2\right) = 1$.

v.) *For every formula* ϕ_1, *for every formula* ϕ_2,

$\mathsf{V}^{\mathsf{M},g}\left((\phi_1 \vee \phi_2)\right) = 1$ *iff* $\mathsf{V}^{\mathsf{M},g}\left(\phi_1\right) = 1$ *or* $\mathsf{V}^{\mathsf{M},g}\left(\phi_2\right) = 1$.

vi.) *For every formula* ϕ_1, *for every formula* ϕ_2,

$\mathsf{V}^{\mathsf{M},g}\left((\phi_1 \rightarrow \phi_2)\right) = 1$ *iff* $\mathsf{V}^{\mathsf{M},g}\left(\phi_1\right) = 0$ *or* $\mathsf{V}^{\mathsf{M},g}\left(\phi_2\right) = 1$.

vii.) *For every formula* ϕ_1, *for every formula* ϕ_2,

$\mathsf{V}^{\mathsf{M},g}\left((\phi_1 \leftrightarrow \phi_2)\right) = 1$ *iff* $\mathsf{V}^{\mathsf{M},g}\left(\phi_1\right) = \mathsf{V}^{\mathsf{M},g}\left(\phi_2\right)$.

viii.) *For every* $v \in \mathsf{Var}$, *for every formula* ϕ,

$\mathsf{V}^{\mathsf{M},g}\left(\forall v\ \phi\right) = 1$ *iff for all* $d \in \mathsf{D}$, $\mathsf{V}^{\mathsf{M},g\overset{d}{v}}\left(\phi\right) = 1$.

ix.) *For every* $v \in \mathsf{Var}$, *for every formula* ϕ,

$\mathsf{V}^{\mathsf{M},g}\left(\exists v\ \phi\right) = 1$ *iff for at least one* $d \in \mathsf{D}$, $\mathsf{V}^{\mathsf{M},g\overset{d}{v}}\left(\phi\right) = 1$.

References

[Ebbinghaus et al., 1992] Ebbinghaus, Heinz-Dieter, Flum, Jörg, and Thomas, Wolfgang 1992. *Einführung in die mathematische Logik*. B.I.-Wissenschaftsverlag, 3rd edition.

[Gamut, 1991a] Gamut, L. T. F. 1991a. *Logic, Language, and Meaning. Volume I. Introduction to Logic*. The University of Chicago Press.

[Gamut, 1991b] Gamut, L. T. F. 1991b. *Logic, Language, and Meaning. Volume II. Intensional Logic and Logical Grammar*. The University of Chicago Press.